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OBLIGATION RULES

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Abstract

We provide a characterization of the obligation rules in the context of minimum cost spanning tree games. We also explore the relation between obligation rules and random order values of the irreducible cost game - it is shown that the later is a subset of the obligation rules. Moreover we provide a necessary and sufficient condition on obligation function such that the corresponding obligation rule coincides with a random order value.

1 Introduction

There is a wide range of economic contexts in which *aggregate costs* have to be allocated amongst individual agents or components who derive the benefits from a common project. A firm has to allocate overhead costs amongst its different divisions. Regulatory authorities have to set taxes or fees on individual users for a variety of services. If several municipalities use a common water supply system, they must reach an agreement on how to share the costs of operating it. In most of these examples, there is no external force such as the market, which determines the allocation of costs. Thus, the final allocation of costs is decided either by mutual agreement or by an *arbitrator* on the basis of some notion of *distributive justice*. The main thrust of this area of research is the axiomatic analysis of allocation rules. Such an axiomatic analysis is supposed to enlighten an *arbitrator* on the possible interpretations of *fairness* while dividing the cost among the participants.

In this paper, we pursue this axiomatic analysis of cost allocation rules for a specific class of cost allocation problems known as *Minimum Cost Spanning*

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Tree Problems or in short *mcstp*. The common feature of these problems is that a group of users has to be connected to a single supplier of some service. For instance, several towns may draw power from a common power plant, and hence have to share the cost of the distribution network. There is a positive cost of connecting each pair of users (towns) as well as a cost of connecting each user (town) to the common supplier (power plant). A cost game arises because cooperation reduces aggregate costs - it may be cheaper for town A to construct a link to town B which is *nearer* to the power plant, rather than build a separate link to the plant. An efficient network must be a *tree* which connects all users to the common supplier. From economic perspective, the main objective here, is to divide the cost of efficient network among its beneficiaries. Following is an example of *mcstp*.

Example 1.1 *See Figure 1, which depicts a minimum cost spanning tree problem. There are three agents* $1, 2$ *and* 3*. The source is denoted by* 0*. These nodes are represented by circles, where the connection between them are represented by straight lines. The numbers represent the cost of each connection. For instance the cost of connection between agent* 1 *and agent* 3 *is* 2*.*

The early literature on minimum cost spanning tree problems mainly focussed on the algorithmic issues of finding an efficient network. Kruskal [1956], Prim [1957] introduced two different versions of the *greedy algorithm* for this purpose. The first game theoretic approach to minimum cost spanning tree problem is due to Bird (1976). Bird, by constructing an allocation rule, proved that core of a minimum cost spanning tree problem is always non-empty. In recent years a number of papers have analyzed this problem from economic perspective and the literature has grown into two different direction. In one hand Granot and Huberman (1984), Kar (2002), Bergantiños and Vidal-Puga (2006a) and Bergantiños and Vidal-Puga (2006b) have discussed various transferable utility games and used them to construct allocation rules, on the other, Norde, Moretti and Tijs (2004), Tijs, Branzei, Moretti and Norde (2006) and Lorenzo-Freire and Lorenzo (2006) have developed a class of allocation methods, known as the obligation rules.

Granot and Huberman (1984) introduced a *TU* game associated with the minimum cost spanning tree problem and analyzed the structure of it's core and nucleolus. Kar (2002) characterized the Shapley value (Shapley (1953)) of this game. Bergantiños and Vidal-Puga (2006a) introduced the *irreducible form* of a minimum cost spanning tree problem and looked at an alternative *TU* game based on the irreducible form. They also characterized the Shapley value of this new games in terms of monotonicity type axiom. Parallel to this literature, Tijs, Branzei, Moretti and Norde (2006) introduced a class of allocation rules for minimum cost spanning tree problems, known as the *obligation rules*. A common feature of the Obligation rules is the fact that players have the possibility to control the cost allocation problem during the construction procedure, i.e. edge by edge following the Kruskal algorithm, of the minimum cost spanning network. Via such a step-by-step cost allocation procedure, players specify how to share the cost of each edge according to a predetermined cost allocation protocol, namely the *obligation functions.* Lorenzo-Freire and Lorenzo (2006) characterized the obligation rules in terms of a restricted *additivity* property.

In this paper we seek to bridge the gap between these two stream of literature. We show that the obligation rules are actually a generalized version of the Shapley value. To be precise, we prove that the random order values (which is itself a generalization of the Shapley value) is a subset of the obligation rules. We also provide a necessary and sufficient condition to distinguish random order values from the obligation rules. and finally characterize the obligation rules with two basic monotonicity properties, namely *population monotonicity* and *strong cost monotonicity.* By focussing on the obligation rules, we try to argue that if one is looking for allocation rules that share the nice properties of the Shapley value but not *anonymous*, then obligation rules are the natural candidates.

In Section 2, we introduce the minimum cost spanning tree problem. In Section 3 and 4, we discuss the random order values and the obligation rules respectively. Section 5 explores the relation between random order values and the obligation rules. Characterization of the obligation rules are provided in Section 6. Section 7 concludes this paper by looking at the probabilistic values for general TU games and their relation with the obligation rules.

2 Minimum cost spanning tree problems

Let $N \subset \mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source.* Usually we take $N = \{1, ..., |N|\}$ where $|N|$ denotes the cardinal of the set N. Let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i. e.* $Pre\,(i, \pi) = \{j \in N \mid \pi\,(j) < \pi\,(i)\}$ .As usual, $\mathcal{R}_+$ denotes the set of non-negative real numbers. Given a set A, let

$$\Delta\,(A) = \left\{ (x_i)_{i \in A} \in \mathcal{R}_+^A : \sum_{i \in A} x_i = 1 \right\}$$

be the simplex in $\mathcal{R}^A$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e* $(i, j) = (j, i)$. We denote the set of all cost matrices over N as $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$. Analogously, given x, $y \in \mathcal{R}^N$, we say $x \leq y$ if $x_i \leq y_i$ for all $i \in N$. A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is the set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix. Given an *mcstp* (N_0, C) , we define the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i,j) \text{ such that } i,j \in N_0\}$. The elements of g are called *arcs*. Given a network g and $i,j \in N_0$, we say that $i,j \in N_0$ are *connected* in g if there exists a sequence of arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$. A *tree* is a network satisfying that for all $i \in N$, there is a unique sequence of arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ connecting i and the source. If t is a tree we usually write $t = \left\{\left(i^0, i\right)\right\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0. Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree t such that $c(t) = \min\{c(t') : t' \text{ is a tree}\}$. It is well-known in the literature of *mcstp* that an *mt* exists, even though it does not necessarily have to be unique. Given an *mcstp* (N_0, C) we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

Given an *mcstp* (N_0, C) and an *mt* t, Bird (1976) defines the *minimal network* (N_0, C^t) associated with t as follows: $c_{ij}^t = \max\limits_{(k,l) \in \bar{t}_{ij}} \{c_{kl}\}$, where $\bar{t}_{ij}$ denotes the unique path in t from i to j. Even though this definition is dependent on the choice of *mt* t, it is independent of the chosen t. Proof of this can be found, for instance, in Aarts and Driessen (1993). The *irreducible form* of an *mcstp* (N_0, C) is defined as the minimal network (N_0, C^*). If (N_0, C^*) is an irreducible problem, then we say that C^* is an *irreducible matrix.* Bergantiños and Vidal-Puga (2006a) proves that (N_0, C) is irreducible if and only if there exists an *mt* t in (N_0, C) satisfying the two following conditions:

$(A1)$ $t^* = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ where $i_0 = 0$.

$(A2)$ Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max\limits_{p<r\leq q} \left\{c_{i_{r-1} i_r}\right\}$.

Example 2.1 *Let* (N_0, C) *be a mcstp as in Example 1.1. The irreducible matrix associated with* (N_0, C) *is depicted in Figure 2.*

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source among them. A *(cost allocation) rule* is a function f such that $f(N_0, C) \in \mathcal{R}^N$ for each *mcstp* (N_0, C) and $\sum\limits_{i \in N} f_i(N_0, C) = m(N_0, C)$. As usually, $f_i(N_0, C)$ represents the cost allocated to agent i. In this paper we focus on the Obligation rules and random order values. We discuss each of these in detail in Section 4 and 3 respectively. In Section 6, we characterize the obligation rules. Following are the axioms, we will be using in this paper.

Strong cost monotonicity (SCM). For all *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$, $f(N_0, C) \leq f(N_0, C')$

SCM says that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off.

SCM is called cost monotonicity in Tijs *et al* (2006) and solidarity in Bergantiños and Vidal-Puga (2006a).

Population monotonicity (PM). For all *mcstp* (N_0, C), $S \subset T \subset N$, and $i \in S$, we have

$$f_i(T_0, C) \leq f_i(S_0, C).$$

PM says that if new agents join a "society" no agent from the "initial society" can be worse off.

The following property is a weak form of anonymity. It says that if two agents are identical in terms of their connection costs then their cost shares must be equal.

Equal Treatment of Equals (ETE). Let (N_0, C) be a *mcstp* such that for some $i, j \in N$, $c_{ki} = c_{kj}$ for all $k \in N_0 \setminus \{i, j\}$. ETE says that $f_i(N_0, C) = f_j(N_0, C)$.

Next property was introduced in Branzei *et al* (2004).

Cone-wise positive linearity (CPL). Let (N_0, C) and (N_0, C') be two *mcstp* satisfying that there exists an order $\sigma : \{(i, j)\}_{i,j \in N_0, i<j} \to \left\{1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that for all $i, j, k, l \in N_0$ satisfying that $\sigma(i, j) \leq \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Thus, for each $x, x' \in \mathcal{R}_+$

$$f(N_0, xC + x'C') = xf(N_0, C) + x'f(N_0, C').$$

This property is an additivity property restricted to some subclass of problems.

It makes no sense to claim additivity in all *mcstp* because $m(N_0, C + C')$ could be different from $m(N_0, C) + m(N_0, C')$. See Bergantiños and Vidal-Puga (2006b) for a detailed discussion on this issue.

Constant share of extra cost $(CSEC)$.

Let (N_0, C), (N_0, C'), (N_0, C^x) and (N_0, C'^x) be a set of *mcstp* satisfying the following conditions:

- For all $i \in N$, $c_{0i} = c_0$, $c'_{0i} = c'_0$, $c^x_{0i} = c_0 + x$, and $c'^x_{0i} = c'_0 + x$ where $x \in \mathcal{R}_+$.
- For all $i, j \in N$, $c_{ij} = c^x_{ij} \leq c_0$ and $c'_{ij} = c'^x_{ij} \leq c'_0$.

Thus,

$$f_i(N_0, C^x) - f_i(N_0, C) = f_i(N_0, C'^x) - f_i(N_0, C').$$

This property is interpreted as follows. A group of agents N faces two problems (N_0, C) and (N_0, C'). In both problems all agents have the same connection cost to the source ($c_{0i} = c_0$ and $c'_{0i} = c'_0$). Moreover, this cost is greater than the connection costs between agents ($c_{ij} \leq c_0$ and $c'_{ij} \leq c'_0$). Under these circumstances, an *mt* implies that any one agent connects directly to the source, and that the rest connect to the source through this agent. Agents agree that the correct solution is f. Assume that an error was made and that the connection cost to the source is x units larger. $CSEC$ states that agents should share this extra cost x in the same way in both problems.

This property is a generalization of the property of Equal Share of Extra Costs ($ESEC$) defined in Bergantiños and Vidal-Puga (2006a). $ESEC$ says that x must be divided equally among agents.

A *game with transferable utility*, briefly a *TU game*, is a pair (N, v) where $v : 2^N \rightarrow \mathcal{R}$ satisfies that $v(\emptyset) = 0$. Based on a *mcstp*, we can define a *TU* game as follows. Given (N_0, C), for each coalition $S \subseteq N$, consider the aggregate cost of a minimum cost spanning tree. That is $v_C(S) = m(S_0, C)$ for all $S \subseteq N$. Kar (2002) axiomatized the Shapley value of v_C. On the other hand Bergantiños and Vidal-Puga (2006a) characterized the Shapley value of a game associated with the irreducible form. This game is defined as, $v_{C^*}(S) = m(S_0, C^*)$ for all $S \subseteq N$. In this paper we will concentrate on the later and explore its relation with the obligation rules.

3 The random order values of the irreducible game

Weber (1988) defines the *random order values* of a *TU* game (N, v). The idea is the following. Agents arrive sequentially. Each agent receives his marginal contribution to his predecessors. To each probability distribution over the set of possible orders, we can define the random order value giving to each agent his expected marginal contribution.

Let $\Delta(\Pi_N)$ be the simplex in $\mathcal{R}^{\Pi_N}$. For each $w = (w_\pi)_{\pi \in \Pi_N} \in \Delta(\Pi_N)$ we define the random order value ROV^w associated with w as follows. Given the *TU* game (N, v) and $i \in N$,

$$ROV_i^w(N, v) = \sum_{\pi \in \Pi_N} w_\pi \left(v\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v\left(Pre\left(i, \pi\right)\right)\right)$$

This definition can be easily rewritten as

$$\begin{aligned} ROV_i^w(N, v) &= \sum_{\pi \in \Pi_N} w_\pi \left[v\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v\left(Pre\left(i, \pi\right)\right)\right] \\ &= \sum_{S \subseteq N \setminus \{i\}} \sum_{\{\pi | Pre(i,\pi) = S\}} w_\pi \left[v\left(S \cup \{i\}\right) - v\left(S\right)\right] \\ &= \sum_{S \subseteq N \setminus \{i\}} \left(\sum_{\{\pi | Pre(i,\pi) = S\}} w_\pi \right) \left[v\left(S \cup \{i\}\right) - v\left(S\right)\right] \end{aligned}$$

We define the family of rules W in *mcstp* as the random order values of the game v_{C^*}. That is, given $w \in \Delta(\Pi_N)$, we define $f^w(N_0, C) = ROV^w(N, v_{C^*})$ for all *mcstp* (N_0, C). Formally,

$$f_i^w(N_0, C) = \sum_{S \subseteq N \setminus \{i\}} \left(\sum_{\{\pi | Pre(i,\pi)=S\}} w_\pi \right) [v_{C^*}(S \cup \{i\}) - v_{C^*}(S)] \quad (1)$$

and

$$W = \{f^w : w \in \Delta(\Pi_N)\}.$$

A well known member of W is the Shapley value. Let us denote this rule by Φ. $\Phi = f^w$, where w is such that all the orders have equal weight. That is $w(\pi) = \frac{1}{|\Delta(\Pi_N)|}$ for all $\pi \in \Delta(\Pi_N)$. Bergantiños and Vidal-Puga (2006a) provides a characterization of Φ, in terms of SCM, PM, and $ESEC$. Following is an example of random order values.

Example 3.1 *Let (N_0, C) be a mcstp as in Example 1.1. The TU game associated with C^* is as follows,*

$v_{C^*}(\{1\}) = v_{C^*}(\{2\}) = 4$, $v_{C^*}(\{3\}) = 6$, $v_{C^*}(\{1,2\}) = 6$, $v_{C^*}(\{1,3\}) = v_{C^*}(\{2,3\}) = 10$, $v_{C^*}(\{1,2,3\}) = 12$.

Let $w(\pi) = \frac{1}{|\Delta(\Pi_N)|}$ for all $\pi \in \Delta(\Pi_N)$. Then, $f_1^w(N_0, C) = f_2^w(N_0, C) = 3$ and $f_3^w(N_0, C) = 6$.

4 Obligation rules

Tijs *et al* (2006) introduce the family of obligation rules in *mcstp*. They prove that each obligation rule satisfy SCM and PM. In this section we present two axiomatic characterizations of obligation rules. In the first one we characterize obligation rules as the only rules satisfying SCM, PM, and CPL. In the second one we characterize obligation rules as the only rules satisfying SCM, PM, and $CSEC$.

We first introduce obligation rules formally. We present this definition in a way a little bit different from Tijs *et al* (2006) in order to adapt it to the objectives of our paper.

Given a network g we define $P(g) = \{T_k(g)\}_{k=1}^{n(g)}$ as the partition of N_0 in *connected components* induced by g. Namely, $P(g)$ is the only partition of N_0 satisfying the following two properties: Firstly, if $i, j \in T_k(g)$, i and j are connected in g Secondly, if $i \in T_k$, $j \in T_l$ and $k \neq l$, i and j are not connected in g.

Given a network g, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs to.

Kruskal (1956) defines an algorithm for computing an *mt*. The idea of the algorithm is to construct a tree by sequentially adding arcs with the lowest cost and without introducing cycles. Formally, Kruskal's algorithm is defined as follows. We start with $A(C) = \{i, j \in N_0 \mid i \neq j\}$ and $g^0(C) = \emptyset$.

Stage 1: Take an arc $(i,j) \in A(C)$ such that $c_{ij} = \min_{(k,l)\in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Now $\left(i^1(C), j^1(C)\right) = (i,j)$, $A(C) = A(C) \setminus \{(i,j)\}$ and $g^1(C) = \left\{\left(i^1(C), j^1(C)\right)\right\}$.

Stage $p+1$. We have defined the sets $A(C)$ and $g^p(C)$. Take an arc $(i,j) \in A(C)$ such that $c_{ij} = \min_{(k,l)\in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Two cases are possible:

1. $g^p(C) \cup \{(i,j)\}$ has a cycle. Go to the beginning of Stage $p+1$ with $A(C) = A(C) \setminus \{(i,j)\}$ and $g^p(C)$ the same.
2. $g^p(C) \cup \{(i,j)\}$ has no cycles. Take $\left(i^{p+1}(C), j^{p+1}(C)\right) = (i,j)$, $A(C) = A(C) \setminus \{(i,j)\}$ and $g^{p+1}(C) = g^p(C) \cup \left\{\left(i^{p+1}(C), j^{p+1}(C)\right)\right\}$. Go to Stage $p+2$.

This process is completed in $|N|$ stages. We say that $g^{|M|}(C)$ is a tree obtained following Kruskal's algorithm. Notice that this algorithm leads to a tree, but that this is not always unique.

When there is not ambiguity we write A, g^p, and (i^p, j^p) instead of $A(C)$, $g^p(C)$, and $(i^p(C), j^p(C))$ respectively.

Given $N \subset \mathcal{N}$, an *obligation function* for N is a map o assigning to each $S \in 2^N \setminus \{\emptyset\}$ a vector $o(S)$ which satisfies the following properties.

o-i) $o(S) \in \Delta(S)$.

o-ii) For each $S, T \in 2^N \setminus \{\emptyset\}$, $S \subset T$ and $i \in S$, $o_i(S) \geq o_i(T)$.

To each obligation function o we can associate an *obligation rule* f^o. The idea is as follows. At each stage of Kruskal's algorithm an arc is added to the network. The cost of this arc will be paid by the agents who benefit from adding this arc. Each of these agents pays the difference between his obligation before the arc is added to the network and after it is added. See Tijs *et al* (2006) for a more detailed discussion.

We now define f^o formally. Given an *mcstp* (N_0, C), let $g^{|N|}$ be a tree obtained applying Kruskal's algorithm to (N_0, C). For each $i \in N$,

$$f_i^o(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p} \left(o_i\left(S\left(P\left(g^{p-1}\right), i\right)\right) - o_i\left(S\left(P\left(g^p\right), i\right)\right)\right)$$

where, by convention, $o_i(T) = 0$ if the source is in T.

We define the family of obligation rules as

$$O = \{f^o : o \text{ is an obligation function}\}$$

Let us clarify the idea of obligation rules with an example.

Example 4.1 *Let* (N_0, C) *be a mcstp as in Example 1.1. An obligation function* o *is as follows,* $o_i(S) = \frac{1}{|S|}$ *for all* $i \in S$ *and for all* $S \subseteq N$. *Let us follow Kruskal's algorithm and compute the cost allocation for* C.
Step 1*: The arc* $(1, 2)$ *is the cheapest. Thus* $\left\{\left(i^1(C), j^1(C)\right)\right\} = \{(1, 2)\}$.
Step 2*: The arc* $(1, 0)$ *is the cheapest among the rest. Thus* $\left\{\left(i^2(C), j^2(C)\right)\right\} = \{(1, 0)\}$.
Step 3*: The arc* $(0, 2)$ *is the cheapest among the rest, but this will form a cycle. Hence we pick the next cheapest arc, which is either* $(1, 3)$ *or* $(2, 3)$. *There is a tie, so let us select* $(1, 3)$. *Thus* $\left\{\left(i^3(C), j^3(C)\right)\right\} = \{(1, 3)\}$.

Now, we can compute the allocation,
$f_1^o(N_0, C) = c_{12}\left[o_1(\{1\}) - o_1(\{1, 2\})\right] + c_{10}\left[o_1(\{1, 2\}) - 0\right] + c_{10}[0 - 0] = 3$
Similarly $f_2^o(N_0, C) = 3$. *Finally,*
$f_3^o(N_0, C) = c_{12}\left[o_3(\{3\}) - o_3(\{3\})\right] + c_{10}\left[o_3(\{3\}) - o_3(\{3\})\right] + c_{10}\left[o_3(\{3\}) - 0\right] = 6$

Next, we provide an alternative representation of obligation functions. This representation will be used later to establish the relationship between W and O.

4.1 Alternative representation of obligation functions

Given $N \subset \mathcal{N}$, an *obligation function* for N is a map d assigning to each $S \in 2^N \setminus \{\emptyset\}$ a vector $d(S)$ which satisfies the following properties,

d-i) $\sum_{i \in N} d_i(N) = 1$. For all $S \subset N$, $\sum_{i \in S} d_i(S) = \sum_{j \notin S} d_j(S \cup j)$.

d-ii) For all $i \in N$, $d_i(N) \geq 0$. For all $S \subset N$, for all $k \in N \setminus S$ and for all $i \in S$, $\sum_{S \subseteq T \subseteq N \setminus k} d_i(T) \geq 0$.

In Theorem 4.1 we will show that above two definitions of an obligation function are equivalent. That is, given an o, one can identify a d and vice versa. We will denote these mappings by $d^*(o)$ and $o^*(d)$ respectively. Let us first describe $d^*(o)$ and $o^*(d)$. We start with some new notations. Suppose o is an obligation function for N. Let $\partial(o)$ (or $\partial(d)$) denotes the set over which o (or d) is an obligation function. That is $\partial(o) = N$. For the rest of this paper, we will use $\partial(o)$ and N interchangeably as per convenience. Let us also define a new mapping o^{-k} from the set $[2^{N \setminus \{k\}} \setminus \{\emptyset\}]$ as follows. $o^{-k}(T) = o(T)$ for all $T \subseteq N \setminus \{k\}$. It is immediate that o^{-k} is an obligation function for $N \setminus \{k\}$. Thus $\partial(o^{-k}) = N \setminus \{k\}$.

Description of $d^*(o)$:
For all $i \in S$, $d_i^*(S, o) = o_i(S)$ when $|\partial(o) \setminus S| = 0$ [1]. Suppose we have already defined $d_i^*(S, o)$, $i \in S$, for all obligation functions o and coalitions S such that

[1]Which simply means, for all $i \in N$, $d_i^*(N, o) = o_i(N)$

$|\partial(o) \setminus S| < m$. Now we define $d_i^*(S, o)$, $i \in S$, for all coalitions $S \subseteq \partial(o)$ such that $|\partial(o) \setminus S| = m$. Choose any $k \in \partial(o) \setminus S$,

$$d_i^*(S, o) = d_i^*(S, o^{-k}) - d_i^*(S \cup \{k\}, o) \tag{2}$$

Note that, $|\partial(o) \setminus S| = m$ implies $|\partial(o^{-k}) \setminus S| < m$ and $|\partial(o) - (S \cup \{k\})| < m$ and hence $d_i^*(S, o^{-k})$ and $d_i^*(S \cup \{k\}, o)$ are well defined.

To complete the description, let us illustrate that d does not depend on the choice of $k \in \partial(o) \setminus S$ either. For $|\partial(o) \setminus S| < 2$, this is vacuously true. We can use induction to prove it in general. Assume that our assertion is correct for all o, $S \subseteq \partial(o)$ such that $|\partial(o) \setminus S| < m$. Let us prove that the same is true for S, such that $|\partial(o) \setminus S| = m$. Since $m \geq 2$, there are at least two agents in $\partial(o) \setminus S$. Suppose $k, l \in \partial(o) \setminus S$. Now, we will show that for all $i \in S$, $d_i^*(S, o^{-k}) - d_i^*(S \cup \{k\}, o) = d_i^*(S, o^{-l}) - d_i^*(S \cup \{l\}, o)$. Since $|\partial(o^{-k}) \setminus S| = m - 1$ and $|\partial(o) \setminus (S \cup \{k\})| = m - 1$, by induction hypothesis $d_i^*(S, o^{-k})$ and $d_i^*(S \cup \{k\}, o)$ do not depend upon the choice of agent, so let us choose l.

$$\begin{aligned}
& d_i^*(S, o^{-k}) - d_i^*(S \cup \{k\}, o) \\
&= \left[d_i^*(S, (o^{-k})^{-l}) - d_i^*(S \cup l, o^{-k})\right] - \left[d_i^*(S \cup k, o^{-l}) - d_i^*(S \cup \{k, l\}, o)\right] \quad \text{[by Equation 2]} \\
&= \left[d_i^*(S, (o^{-k})^{-l}) - d_i^*(S \cup k, o^{-l})\right] - \left[d_i^*(S \cup l, o^{-k}) - d_i^*(S \cup \{k, l\}, o)\right] \\
&= \left[d_i^*(S, (o^{-l})^{-k}) - d_i^*(S \cup k, o^{-l})\right] - \left[d_i^*(S \cup l, o^{-k}) - d_i^*(S \cup \{k, l\}, o)\right] \quad [(o^{-l})^{-k} = (o^{-k})^{-l}\ ] \\
&= d_i^*(S, o^{-l}) - d_i^*(S \cup \{l\}, o) \qquad \text{[by Equation 2 and induction step]}
\end{aligned}$$

Description of $o^*(d)$:

For all $S \subseteq \partial(d)$, for all $i \in S$,

$$o_i^*(S, d) = \sum_{S \subseteq T \subseteq \partial(d)} d_i(T) \tag{3}$$

Theorem 4.1 *: o and d are equivalent. That is,*

1. $o^*(d^*(o)) = o$.

2. $d^*(o^*(d)) = d$.

3. $d^*(o)$ *satisfies d-i and d-ii.*

4. $o^*(d)$ *satisfies o-i and o-ii.*

Proof. See Appendix 8.1.

Let us now provide an example to illustrate this equivalence.

Example 4.2 *: $N = \{1, 2, 3, 4\}$. An obligation function o is described as follows, $o_i(S) = \frac{1}{|S|}$ for all $i \in S$ for all $S \subseteq N$. Alternative representation of this obligation function is as follows. For all $i \in S$,*

$$d_i(S) = \begin{cases} \frac{1}{4} & \text{if } |S| = 1, 4 \\ \frac{1}{12} & \text{if } |S| = 2, 3 \end{cases}$$

One can verify that $d^*(o) = d$ and $o^*(d) = o$. We illustrate these for a couple of subcoalitions, the rest is left to the readers.

$$\begin{aligned}
&d_1^*(12, o)\\
&\quad= d_1^*(\{1,2\}, o^{-4}) - d_1^*(\{1,2,4\}, o)\\
&\quad= [d_1^*(\{1,2\}, (o^{-4})^{-3}) - d_1^*(\{1,2,3\}, o^{-4})] - [d_1^*(\{1,2,4\}, o^{-4}) - d_1^*(\{1,2,3,4\}, o)]\\
&\quad= [o_1(\{1,2\}) - o_1(\{1,2,3\})] - [o_1(\{1,2,4\}) - o_1(\{1,2,3,4\})]\\
&\quad= \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{4}\right) = \frac{1}{12}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
o_1^*(\{1,2\}, d) &= o_1^*(\{1,2\}, d) + o_1^*(\{1,2,3\}, d) + o_1^*(\{1,2,4\}, d) + o_1^*(\{1,2,3,4\}, d)\\
&= \left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{4}\right) = \frac{1}{2}
\end{aligned}$$

The next example illustrates that it is not necessary to have $d_i(S) \geq 0$ for all $i \in S$ and for all $S \subseteq N$. This observation will play a crucial role while exploring the relationship between W and O.

Example 4.3 *: $N = \{1,2,3,4\}$. Consider an obligation function $\hat{o}$, as in Example 4.2 except for the set $\{1,2,3\}$.*

$$\hat{o}_i(\{1,2,3\}) = \begin{cases} \frac{1}{2} & \textit{if } i = 1 \\ \frac{1}{4} & \textit{Otherwise} \end{cases}$$

It is straightforward to check that o satisfies o-i and o-ii. Note that

$$\begin{aligned}
d_1^*(12, \hat{o}) &= [\hat{o}_1(\{1,2\}) - \hat{o}_1(\{1,2,3\})] - [\hat{o}_1(\{1,2,4\}) - \hat{o}_1(\{1,2,3,4\})]\\
&= \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\right) = -\frac{1}{12}
\end{aligned}$$

Since o and d are equivalent, for the rest of this paper, instead of $d^*(o)$ and $o^*(d)$, we will simply write $d(o)$ and $o(d)$ respectively.

4.2 Alternative representation of obligation rules

The alternative representation of obligation functions, that is d, gives rise to a new definition of the obligation rules. This representation will reveal the essential similarity between the obligation rules and the random order values. In the process, a natural interpretation of d also emerges. The main result of this section is the following theorem, which rewrites the obligation rules in terms of d.

Theorem 4.2 *: If $f^o \in O$, then for all $i \in N$, for all mcstp (N_0, C)*

$$f_i^o(N_0, C) = \sum_{S \subseteq N \backslash \{i\}} d_i(N \backslash S, o)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right] \tag{4}$$

Proof. See Appendix 8.2.

A quick look at Equation 1 and Equation 4 confirms that they are closely related. Before proceeding to the next section, where this relation will be investigated thoroughly, let us end the current discussion with an interpretation of d. To this end, the following result will be extremely helpful.

Proposition 4.3 *Let o be an obligation function. Then*

$$\sum_{S \subseteq N \backslash \{i\}} d_i(N \backslash S, o) = 1$$

Proof. The proof is straightforward.

$$\sum_{S \subseteq N \backslash \{i\}} d_i(N \backslash S, o) = \sum_{i \in T} d_i(T, o) = \sum_{\{i\} \subseteq T \subseteq N} d_i(T, o) = o_i(\{i\}, d)$$

The last equality follows from Equation 3. By Theorem 4.1, $o(d(o)) = o$. Thus,

$$\sum_{S \subseteq N \backslash \{i\}} d_i(N \backslash S, o) = o_i(\{i\}, d) = o_i(\{i\}) = 1.$$

Proposition 4.3 tells us that $\{d_i(N \setminus S, o) \mid S \subseteq N \setminus \{i\}\}$ acts as an weight system for $f_i^o(N_0, C)$ in Equation 4. In the same spirit, Weber (1988) introduced a class of '*marginalist values*' for transferrable utility games. In Section 7 we will formally introduce marginalist values and farther discuss their relation with obligation rules.

5 Relation between W and O

In the next theorem we study the relationship between W and O. The first part shows that obligation rules are nothing but generalized random order values. Next, we identify a necessary and sufficient condition to distinguish between obligation rules and random order values. In part (*iii*) we propose an obligation rule and show that it does not belong to the set of random order values when population size is strictly greater than 3. If population is less than or equal to 3, then random order values and obligation rules coincide. This is illustrated in corollary 5.1 as an immediate consequence of part (*ii*).

Theorem 5.1 *: W and O are related as follows,* *i)* $W \subseteq O$.

ii) *Let o be an obligation function.* $f^o \in W \Leftrightarrow d_i(S,o) \geq 0$ *for all* $i \in S$ *and for all* $S \subseteq N$.

iii) *If* $|N| > 3$, *then* $W \subset O$.

Proof. See Appendix 8.3.

Corollary 5.1 *If* $|N| \leq 3$, *then* $W = O$.

Proof: Suppose $N = \{1,2,3\}$ and o is an obligation function on N. By d-ii $d_i(N,o) \geq 0$ for all $i \in N$. Now suppose $S = \{1,2\}$. Let us show that $d_1(\{1,2\},o) \geq 0$. The proof is exactly the same for all S with $|S| = 2$ and all $i \in S$.

$$d_1(\{1,2\},o) = [d_1(\{1,2\},o^{-3}) - d_1(\{1,2,3\},o)] = [o_1(\{1,2\}) - o_1(\{1,2,3\})] \geq 0$$

The last inequality follows from o-ii. Next we show that $d_1(\{1\},o) \geq 0$. Again the proof is exactly the same for all S with $|S| = 1$.

$$\begin{aligned}
d_1(\{1\},o) &= [d_1\left(\{1\},o^{-2}\right) - d_1(\{1,2\},o)] \\
&= [d_1\left(\{1\},(o^{-2})^{-3}\right) - d_1(\{1,3\},o^{-2})] - d_1(\{1,2\},o) \\
&= [o_1(\{1\}) - o_1(\{1,3\})] - [o_1(\{1,2\}) - o_1(\{1,2,3\})] \\
&= [1 - o_1(\{1,3\})] - [1 - o_2(\{1,2\})] + [1 - o_2(\{1,2,3\}) - o_3(\{1,2,3\})] \\
&= [o_3(\{1,3\}) + o_2(\{1,2\})] - [o_2(\{1,2,3\}) + o_3(\{1,2,3\})] \\
&= [o_2(\{1,2\}) - o_2(\{1,2,3\})] + [o_3(\{1,3\}) - o_3(\{1,2,3\})] \geq 0
\end{aligned}$$

Thus when $|N| = 3$, for all $S \subseteq N$, and for all $i \in S$, $d_i(S,o) \geq 0$. By part (*iii*) of Theorem 5.1, $f^o \in W$ implying $O \subseteq W$. Hence by part (*i*) of Theorem 5.1, $W = O$. When $|N| \leq 2$ the proof is similar and thus omitted.

We say that $\bar{o}$ is a *simple* obligation function if for each $S \subset N$ and $i \in S$, $\bar{o}_i(S)$ is either 1 or 0. Let SO denote the family of all simple obligation functions.

Consider the family of obligation rules induced by obligation functions which are a convex combination of simple obligation functions. Next corollary says that this family is just W.

Corollary 5.2 $f \in W$ *if and only if* $f = f^{o^{\varpi}}$ *where for each* $S \subset N$ *and* $i \in S$, $o_i^{\varpi}(S) = \sum_{\bar{o} \in SO} \varpi_{\bar{o}} \bar{o}_i(S)$ *and* $(w_{\bar{o}})_{\bar{o} \in SO} \in \Delta(SO)$.

Proof: See Appendix 8.4

The idea of the proof is quite simple. Given a simple obligation function we can associate an order $\pi \in \Pi_N$ and vice versa. Now it is easy to prove that W is the subfamily of O generated by convex hull of simple obligation functions.

6 Axiomatization of O

In the next theorem we give two axiomatic characterizations of obligation rules.

Theorem 6.1 *Let f be a rule in mcstp.*
(a) *f satisfies SCM, PM, and CPL if and only if $f \in O$.*
(b) *f satisfies SCM, PM, and $CSEC$ if and only if $f \in O$.*

Proof. See Appendix 8.5.
.

Remark 6.1 *The properties used in Theorem 6.1 are independent. See Appendix 8.6 for a proof.*

Bergantiños and Vidal-Puga (2006b) introduce the property of Restricted Additivity (RA). They prove that RA is stronger than CPL, *i.e.* if a rule f satisfies RA, f also satisfies CPL. Lorenzo-Freire and Lorenzo (2006) characterize obligation rules as the only rules satisfying RA and SCM. However, RA and CPL behaves in a different way. SCM is a consequence of RA and PM, whereas SCM is independent of CPL and PM. Moreover, the proof of the uniqueness part in both results is completely different. Based on our axiomatization of obligation rules, we present an alternative characterization of Φ.

Corollary 6.1 *Let f be a rule in mcstp. f satisfies SCM, PM, and $CSEC$ and ETE if and only if $f = \Phi$.*

Proof: See Appendix 8.7

7 Conclusion

In this paper, we have tried to bridge the gap between the two streams of literature, that have evolved recently, on minimum cost spanning tree games. We have explored the structural relation between the obligation rules and the random order values and established that random order values are a subset of the obligation rules. In this section we compare our results to Weber (1988). The following definitions are due to Weber.

A value f is called a *marginalistic value* if for each $i \in N$ there exists $p^i \in \mathcal{R}^{\{S:S\subset N\setminus\{i\}\}}$ such that $\sum\limits_{S\subset N\setminus\{i\}} p^i(S) = 1$ and

$$f_i(N,v) = \sum_{S\subset N\setminus\{i\}} p^i(S)\left(v(S\cup\{i\}) - v(S)\right)$$

A value f is called a *probabilistic value* if for each $i \in N$ there exists a probability $p^i \in \Delta(S : S \subset N \backslash \{i\})$ such that for all TU game (N, v), $f_i(N, v)$ is the expected marginal contribution of i with respect to p^i. Namely,

$$f_i(N, v) = \sum_{S \subset N \backslash \{i\}} p^i(S)\left(v(S \cup \{i\}) - v(S)\right)$$

Usually, we denote by f^p the probabilistic value induced by $p = \{p^i\}_{i \in N}$ ..

Weber showed that the marginalistic values are the only rules to satisfy *linearity* and *dummy* axiom over $\mathcal{SU}$. He also proved that on top of *linearity* and *dummy*, if a value satisfy *monotonicity* axiom then it must be a probabilistic value.

A value ϕ satisfies *linearity* if $\phi(v_1+v_2) = \phi(v_1)+\phi(v_2)$ and $\phi(c.v) = c.\phi(v)$, where (N, v_1), (N, v_2), (N, v) are TU games and c is a scalar.

Suppose a TU game (N, v) is such that for all $S \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) = v(S) + v(\{i\})$. A value ϕ satisfies *dummy* if $\phi_i(v) = v(\{i\})$.

Suppose a TU game (N, v) is such that for all $S \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) - v(S) \geq 0$. A value ϕ satisfies *monotonicity* if $\phi_i(v) \geq 0$.

Note that neither marginalistic values nor probabilistic values demand *efficiency*, that is $\sum_{i \in N} \phi_i(v) = v(N)$. The parallel between our results and Weber is captured in the following theorem.

Let the class of marginastic values for *mcstp* is defined as follows. $MV = \{f^p \mid p^i \in \mathcal{R}^{\{S : S \subset N \backslash \{i\}\}}$ and $\sum\limits_{S \subset N \backslash \{i\}} p^i(S) = 1\}$ where

$$f_i(N_0, C) = \sum_{S \subset N \backslash \{i\}} p^i(S)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right]$$

Similarly the class of probabilistic values for *mcstp* will be denoted as, $PV = \{f^p \mid p^i \in \Delta(S : S \subset N \backslash \{i\})\}$ where

$$f_i(N_0, C) = \sum_{S \subset N \backslash \{i\}} p^i(S)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right]$$

Theorem 7.1 *(a) $O \subseteq MV$. Moreover, for all $i \in N$ and all $S \subset N \backslash \{i\}$, $p^i(S) = d_i(N \backslash S, o)$.*
(b) $O \cap PV = W$

This is an immediate consequence of Theorem 5.1.

In this paper we have emphasized the importance of the obligation rules for minimum cost spanning tree problems and explored its link with other standard allocation methods. A future research agenda will be to define and characterize obligation rules for the general TU games.

8 Appendix

We prove the results stated in the paper.

8.1 Proof of Theorem 4.1

We prove this result in several steps. These are as follows,

Step 1: $o^*(d^*(o)) = o$.
Step 2: $d^*(o^*(d)) = d$.
Step 3: $d^*(o)$ satisfies d-i.
Step 4: $d^*(o)$ satisfies d-ii.
Step 5: $o^*(d)$ satisfies o-i.
Step 6: $o^*(d)$ satisfies o-ii.

Proof of step 1: We want to show that for all $S \subseteq N$, for all $i \in S$, $o_i^*(S, d^*(o)) = o_i(S)$. This is trivially true when $|\partial(o) \setminus S| = 0$, because by Equation 3 and the fact that $\partial(d^*(o)) = \partial(o)$, we have, $o_i^*(\partial(o), d^*(o)) = \sum_{\partial(o) \subseteq T \subseteq \partial(d^*(o))} d_i^*(T, o) = d_i^*(\partial(o), o) = o_i(\partial(o))$. We will use induction on $|\partial(o) \setminus S|$ to prove this result. Suppose our assertion is correct for all S, for all $i \in S$, such that $|\partial(o) \setminus S| < \kappa$. We will show that the same is true when $|\partial(o) \setminus S| = \kappa$. Without loss of generality assume that $k \notin S$.

By Equation 3,

$$o_i^*(S, d^*(o)) = \sum_{S \subseteq T \subseteq \partial(o)} d_i^*(T, o) = \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T, o) + \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T \cup \{k\}, o)$$

By Equation 2, this can be written as follows,

$$\begin{aligned}
& o_i^*(S, d^*(o)) \\
= & \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T, o) + \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T \cup \{k\}, o) \\
= & \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} \left[d_i^*(T, o^{-k}) - d_i^*(T \cup \{k\}, o) \right] + \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T \cup \{k\}, o) \\
= & \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T, o^{-k}) - \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T \cup \{k\}, o) + \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T \cup \{k\}, o) \\
= & \sum_{S \subseteq T \subseteq [\partial(o) \setminus \{k\}]} d_i^*(T, o^{-k}) = o_i^*(S, d^*(o^{-k})) = o_i^{-k}(S) \quad \text{(by induction hypothesis)} \\
= & o_i(S)
\end{aligned}$$

Proof of step 2: We want to prove that for all $S \subseteq N$, for all $i \in S$, $d_i^*(S, o^*(d)) = d_i(S)$. This is trivially true when $|\partial(d) \setminus S| = 0$, because

$d_i^*(\partial(d), o^*(d)) = o_i^*(\partial(d), d) = \sum\limits_{\partial(d) \subseteq T \subseteq \partial(d)} d_i(T) = d_i(\partial(d))$. We will use induction on $|\partial(o) \setminus S|$ to prove this result. Suppose our assertion is correct for all S, for all $i \in S$, such that $|\partial(o) \setminus S| < \kappa$. We will show that the same is true when $|\partial(o) \setminus S| = \kappa$.

From step 1, we have, $o^*(d^*(o)) = o$. Letting $o = o^*(d)$, we get $o^*(d^*(o^*(d))) = o^*(d)$. That is for all S, for all $i \in S$, $o_i^*(S, d^*(o^*(d))) = o_i^*(S, d)$. By Equation 3, this implies

$$\begin{aligned} & \sum_{S \subseteq T \subseteq \partial(d)} d_i^*(T, o^*(d)) = \sum_{S \subseteq T \subseteq \partial(d)} d_i(T) \\ \Rightarrow \quad & d_i^*(S, o^*(d)) + \sum_{S \subset T \subseteq \partial(d)} d_i^*(T, o^*(d)) = d_i(S) + \sum_{S \subset T \subseteq \partial(d)} d_i(T) \\ \Rightarrow \quad & d_i^*(S, o^*(d)) = d_i(S) \end{aligned}$$

The last step follows from the induction step, which implies $d_i^*(T, o^*(d)) = d_i^*(T)$, for all T such that $S \subset T \subseteq \partial(d)$.

Proof of step 3: It is easy to check that $\sum\limits_{i \in N} d_i^*(\partial(o), o) = \sum\limits_{i \in N} o_i(\partial(o)) = 1$.

We will use induction to prove $\sum\limits_{i \in S} d_i^*(S, o) = \sum\limits_{j \notin S} d_j^*(S \cup \{j\}, o)$ for all $S \subset \partial(o)$. Let us first show that the above is true when $|\partial(o) \setminus S| = 1$. Without loss of generality we can assume that $\partial(o) \setminus S = \{k\}$. By Equation 2, $\sum\limits_{i \in S} d_i^*(S, o) = \sum\limits_{i \in S} d_i^*(S, o^{-k}) - \sum\limits_{i \in S} d_i^*(S \cup \{k\}, o) = \sum\limits_{i \in \partial(o^{-k})} d_i^*(\partial(o^{-k}), o^{-k}) - \sum\limits_{i \in \partial(o) \setminus \{k\}} d_i^*(\partial(o), o)$ (because $\partial(o^{-k}) = S$ and $\partial(o) = S \cup \{k\}$). Thus by the first result of step 3, $\sum\limits_{i \in S} d_i^*(S, o) = 1 - [1 - d_k^*(\partial(o), o)] = d_k^*(\partial(o), o) = d_k^*(S \cup \{k\}, o) = \sum\limits_{j \notin S} d_j^*(S \cup \{j\}, o)$.

Now, suppose that we have proved this result for all S such that $|\partial(o) \setminus S| < \kappa$. We will prove the same when $|\partial(o) \setminus S| = \kappa$. Without loss of generality, assume that $k \in \partial(o) \setminus S$. By Equation 2, $\sum\limits_{i \in S} d_i^*(S, o) = \sum\limits_{i \in S} d_i^*(S, o^{-k}) - \sum\limits_{i \in S} d_i^*(S \cup \{k\}, o)$. Since $|\partial(o^{-k}) \setminus S| < \kappa$ and $|\partial(o) - (S \cup k)| < \kappa$, by the induction hypothesis,

$$\begin{aligned} & \sum_{i \in S} d_i^*(S, o^{-k}) - \sum_{i \in S} d_i^*(S \cup \{k\}, o) \\ = \quad & \left[\sum_{i \in S} d_i^*(S, o^{-k}) - \sum_{i \in S \cup k} d_i^*(S \cup \{k\}, o) \right] + d_k^*(S \cup \{k\}, o) \\ = \quad & \left[\sum_{j \in \partial(o) \setminus (S \cup k)} d_j^*(S \cup \{j\}, o^{-k}) - \sum_{i \in \partial(o) \setminus (S \cup k)} d_j^*(S \cup \{k, j\}, o) \right] + d_k^*(S \cup \{k\}, o) \end{aligned}$$

Thus by Equation 2,

$$\sum_{i \in S} d_i^*(S, o)$$

$$
\begin{aligned}
&= \sum_{j\in\partial(o)\setminus(S\cup k)} \big[d_j^*(S\cup\{j\},o^{-k}) - d_j^*(S\cup\{k,j\},o)\big] + d_k^*(S\cup\{k\},o)\\
&= \sum_{j\in\partial(o)\setminus(S\cup k)} d_j^*(S\cup\{j\},o) + d_k^*(S\cup\{k\},o) = \sum_{j\notin S} d_j^*(S\cup\{j\},o)
\end{aligned}
$$

This completes step 3.

Proof of step 4: We know that for all $i\in\partial(o)$, $d_i^*(\partial(o),o) = o_i(\partial(o)) \geq 0$ (by o-i). For all $S\subset\partial(o)$, for all $i\in S$ and for all $k\in\partial(o)\setminus S$

$$
\begin{aligned}
\sum_{S\subseteq T\subseteq[\partial(o)\setminus\{k\}]} d_i^*(T,o) &= \sum_{S\subseteq T\subseteq\partial(o)} d_i^*(T,o) - \sum_{S\subseteq T\subseteq[\partial(o)\setminus\{k\}]} d_i^*(T\cup\{k\},o)\\
&= o_i^*(S,d^*(o)) - o_i^*(S\cup\{k\},d^*(o)) \quad \text{(by Equation 3)}\\
&= o_i(S) - o_i(S\cup\{k\}) \quad \text{(by step 1)}\\
&\geq 0 \quad \text{(by o-ii)}
\end{aligned}
$$

Proof of step 5: We need to prove that for all $S\subseteq\partial(d)$, $o^*(S,d)\in\Delta(S)$. We will show it in two parts. First, we show that $o_i^*(S,d)\geq 0$ for all $i\in S$. If $S=\partial(d)$, this is trivially true, because $o_i^*(\partial(d),d) = d_i(\partial(d))$ (by Equation 3)≥ 0 (by d-ii). We will use induction on $|\partial(d)\setminus S|$. Suppose our assertion is correct for all S such that $|\partial(d)\setminus S| < \kappa$. Let us show the same when $|\partial(d)\setminus S| = \kappa$.

$$
\begin{aligned}
o_i^*(S,d) &= \sum_{S\subseteq T\subseteq\partial(d)} d_i(T) \quad \text{(by Equation 3)}\\
&= \sum_{S\subseteq T\subseteq[\partial(d)\setminus\{k\}]} d_i(T) + \sum_{S\subseteq T\subseteq[\partial(d)\setminus\{k\}]} d_i(T\cup\{k\})\\
&= \sum_{S\subseteq T\subseteq[\partial(d)\setminus\{k\}]} d_i(T) + o_i^*(S\cup\{k\},d) \quad \text{(by Equation 3)}\\
&\geq 0
\end{aligned}
$$

The last step follows from d-ii, which implies $\sum\limits_{S\subseteq T\subseteq[\partial(d)\setminus\{k\}]} d_i(T)\geq 0$ and the induction step, which implies $o_i^*(S\cup\{k\},d)\geq 0$.

Now, let us show that $\sum\limits_{i\in S} o_i^*(S,d) = 1$. Again, this is easy to prove when $S=\partial(d)$. Using d-i and Equation 3, we get, $\sum\limits_{i\in\partial(d)} o_i^*(\partial(d),d) = \sum\limits_{i\in\partial(d)} d_i(\partial(d)) = 1$. Now, consider any $S\subset N$. By Equation 3,

$$
\begin{aligned}
\sum_{i\in S} o_i^*(S,d) &= \sum_{i\in S}\sum_{S\subseteq T\subseteq\partial(d)} d_i(T)\\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq 1\}}\sum_{i\in S} d_i(T) + \sum_{i\in S} d_i(S)\\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq 1\}}\sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=0\}}\sum_{i\in T} d_i(T)
\end{aligned}
$$

If we can prove that for all l, such that $1 \leq l \leq (|\partial(d) \setminus S| - 1)$

$$\sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq l\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=(l-1)\}} \sum_{i\in T} d_i(T)$$
$$= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq(l+1)\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=l\}} \sum_{i\in T} d_i(T) \quad (5)$$

then by repeated use Equation 3, we can conclude

$$\begin{aligned}
&\sum_{i\in S} o_i^*(S,d) \\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=|\partial(d)\setminus S|\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=|\partial(d)\setminus S|-1\}} \sum_{i\in T} d_i(T) \\
&= \sum_{i\in S} d_i(\partial(d)) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|\partial(d)\setminus T|=1\}} \sum_{i\in T} d_i(T) \\
&= \sum_{i\in S} d_i(\partial(d)) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|\partial(d)\setminus T|=1\}} d_{\partial(d)\setminus T}(\partial(d)) \quad \text{(by d-i)} \\
&= \sum_{i\in S} d_i(\partial(d)) + \sum_{i\in\partial(d)\setminus S} d_i(\partial(d)) = \sum_{i\in\partial(d)} d_i(\partial(d)) = 1
\end{aligned}$$

To conclude step 5, now we will prove Equation 3. First, note that

$$\begin{aligned}
&\sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=(l-1)\}} \sum_{i\in T} d_i(T) \\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=(l-1)\}} \sum_{i\in\partial(d)\setminus T} d_i(T\cup\{i\}) \quad \text{(by d-i)} \\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=(l-1)\}} \sum_{\{R|T\subseteq R\subseteq\partial(d)||R\setminus T|=1\}} d_{R\setminus T}(R) \\
&= \sum_{\{R|S\subseteq R\subseteq\partial(d),|R\setminus S|=l\}} \sum_{\{T|S\subseteq T\subseteq R,|R\setminus T|=1\}} d_{R\setminus T}(R) \\
&= \sum_{\{R|S\subseteq R\subseteq\partial(d),|R\setminus S|=l\}} \sum_{i\in R\setminus S} d_i(R)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq l\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=(l-1)\}} \sum_{i\in T} d_i(T) \\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq l+1\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=l\}} \sum_{i\in S} d_i(T) \\
&\qquad + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=l\}} \sum_{i\in T\setminus S} d_i(T) \\
&= \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|\geq l+1\}} \sum_{i\in S} d_i(T) + \sum_{\{T|S\subseteq T\subseteq\partial(d),|T\setminus S|=l\}} \sum_{i\in T} d_i(T)
\end{aligned}$$

This proves Equation 3 and completes step 5.

Proof of step 6: We want to prove that for all $S \subset T \subseteq \partial(d)$ and for all $i \in S$, $o_i^*(S,d) \geq o_i^*(T,d)$. It is enough to show that $o_i^*(S,d) \geq o_i^*(S \cup \{k\}, d)$, which then can be used repeatedly to complete this step. Now,

$$\begin{aligned} o_i^*(S,d) - o_i^*(S \cup \{k\}, d) &= \sum_{S \subseteq T \subseteq \partial(d)} d_i(T) - \sum_{[S \cup \{k\}] \subseteq T \subseteq \partial(d)} d_i(T) \\ &= \sum_{S \subseteq T \subseteq [\partial(d) \setminus \{k\}]} d_i(T) \\ &\geq 0 \quad \text{(by d-ii)} \end{aligned}$$

8.2 Proof of Theorem 4.2

First, let us introduce some new notations and lemmata, which will be used later in the main body of proof.

Lemma 8.1 *If (N_0, C) is an irreducible mcstp, $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ is the mt satisfying $(A1)$ and $(A2)$, and $S = \left\{i_{s(1)}, ..., i_{s(|S|)}\right\}$, then*

(a) $v_C(S) = \sum_{q=1}^{|S|} c_{i_{s(q-1)} i_{s(q)}}$ where we denote $s(0) = 0$.

(b) $v_C(S) - v_C\left(S \setminus \left\{i_{s(p)}\right\}\right) = \min\left\{c_{i_{s(p-1)} i_{s(p)}}, c_{i_{s(p)} i_{s(p+1)}}\right\}$ if $p < |S|$ and $v_C(S) - v_C\left(S \setminus \left\{i_{s(|S|)}\right\}\right) = c_{i_{s(|S|-1)} i_{s(|S|)}}$.

Proof of Lemma 8.1: The proof of this result appears in Bergantiños and Vidal-Puga (2006a).

Suppose g is a network on N_0. A *simple mcstp* (N_0, C^g) induced by the network g is defined as follows, $c_{ij}^g = 1$ if $(i,j) \in g$ and $c_{ij}^g = 0$ otherwise.

Lemma 8.2 *For each mcstp (N_0, C), there exists a family $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ of simple mcstp and a family $\{x^q\}_{q=1}^{\tau(C)}$ of non-negative real numbers satisfying two conditions:*

(1) $C = \sum_{q=1}^{\tau(C)} x^q C^{g^q}$

(2) There exists an order $\sigma : \{(i,j)\}_{i,j \in N_0} \to \left\{1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that given $\{i,j,k,l\} \subseteq N_0$ with $\sigma(i,j) \leq \sigma(k,l)$ we have that $c_{ij} \leq c_{kl}$ and $c_{ij}^{g^q} \leq c_{kl}^{g^q}$ for each $q \in \{1, ..., \tau(C)\}$.

Proof of Lemma 8.2: The proof of this result appears in Norde *et al* (2004).

Lemma 8.3 *For each mcstp* (N_0, C)*, there exists a family* $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ *of simple mcstp and a family* $\{x^q\}_{q=1}^{\tau(C)}$ *of non-negative real numbers such that*

(1) $C^* = \sum_{q=1}^{\tau(C)} x^q \left(C^{g^q}\right)^*$

(2) *For all* $S \subseteq N$, $v_{C^*}(S) = \sum_{q=1}^{\tau(C)} x^q v_{\left(C^{g^q}\right)^*}(S)$

Proof of Lemma 8.3: From Lemma 8.2, we know that given a *mcstp* (N_0, C), there exists a family $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ of simple *mcstp* and a family $\{x^q\}_{q=1}^{\tau(C)}$ of non-negative real numbers satisfying, $C = \sum_{q=1}^{\tau(C)} x^q C^{g^q}$ and $\{c_{ij} \leq c_{kl} \Leftrightarrow c_{ij}^{g^q} \leq c_{kl}^{g^q}\}$ for all $\{i, j, k, l\} \subset N_0$ and for all $q \in \{1, ..., \tau(C)\}$. Let us use Prim's algorithm on C and C^{g^q} for all $q \in \{1, ..., \tau(C)\}$. Since cost of the edges follow the same order over these cost matrices, they must have the same minimum cost spanning trees. Let us pick one of these minimum cost spanning trees (say t) and calculate C^* and $\left(C^{g^q}\right)^*$. Take any $\{k, l\} \subseteq N_0$. $\bar{t}_{kl}$ denotes the unique path from k to l on t. Now,

$$
\begin{aligned}
c_{kl}^* &= \max_{(kl) \in \bar{t}_{kl}} c_{ij} \\
&= \max_{(kl) \in \bar{t}_{kl}} \left[\sum_{q=1}^{\tau(C)} x^q c_{ij}^{g^q}\right] \qquad \text{(by (1) of Lemma 8.2)} \\
&= \sum_{q=1}^{\tau(C)} x^q \left[\max_{(kl) \in \bar{t}_{kl}} c_{ij}^{g^q}\right] \qquad \text{(by (2) of Lemma 8.2)} \\
&= \sum_{q=1}^{\tau(C)} x^q \left(c_{kl}^{g^q}\right)^* \qquad (6)
\end{aligned}
$$

Therefore $C^* = \sum_{q=1}^{\tau(C)} x^q \left(C^{g^q}\right)^*$. One can also easily check that $\{c_{ij}^* \leq c_{kl}^* \Leftrightarrow \left(c_{ij}^{g^q}\right)^* \leq \left(c_{kl}^{g^q}\right)^*\}$ for all $\{i, j, k, l\} \subset N_0$ and for all $q \in \{1, ..., \tau(C)\}$.

Now by Lemma 8.1, (N_0, C^*) has an *mt*, $t^* = \{(i_{p-1}, i_p)\}_{p=1}^n$. Since cost of the edges follow the same order over C^* and $\left(C^{g^q}\right)^*$, t^* is also an *mt* for $(N_o, \left(C^{g^q}\right)^*)$ where $q \in \{1, ..., \tau(C)\}$. Suppose, $S = \left\{i_{s(1)}, ..., i_{s(|S|)}\right\}$. Then,

$$
\begin{aligned}
v_{C^*}(S) &= \sum_{q=1}^{|S|} c_{i_{s(q-1)} i_{s(q)}}^* \qquad \text{(by Lemma 8.1)} \\
&= \sum_{q=1}^{|S|} \sum_{q=1}^{\tau(C)} x^q \left(c_{i_{s(q-1)} i_{s(q)}}^{g^q}\right)^* \qquad \text{(by Equation 6)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{q=1}^{\tau(C)} x^q \left[\sum_{q=1}^{|S|} \left(c^{g^q}_{i_{s(q-1)} i_{s(q)}} \right)^* \right] \\
&= \sum_{q=1}^{\tau(C)} x^q v_{\left(C^{g^q}\right)^*}(S) \qquad \text{(by Lemma 8.1)}
\end{aligned}
$$

Lemma 8.4 *For each mcstp (N_0, C), there exists a family $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ of simple mcstp and a family $\{x^q\}_{q=1}^{\tau(C)}$ of non-negative real numbers such that $f^o(N_0, C) = \sum_{q=1}^{\tau(C)} x^q f^o(N_0, C^{g^q})$.*

Proof of Lemma 8.4: From Lemma 8.2, we know that given a *mcstp* (N_0, C), there exists a family $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ of simple *mcstp* and a family $\{x^q\}_{q=1}^{\tau(C)}$ of non-negative real numbers satisfying, $C = \sum_{q=1}^{\tau(C)} x^q C^{g^q}$ and $\{c_{ij} \leq c_{kl} \Leftrightarrow c^{g^q}_{ij} \leq c^{g^q}_{kl}\}$ for all $\{i,j,k,l\} \subset N_0$ and for all $q \in \{1, ..., \tau(C)\}$. Now, we will use Kruskal's algorithm on C and C^{g^q} for all $q \in \{1, ..., \tau(C)\}$. Since cost of the edges follow the same order over these cost matrices, Kruskal's algorithm will also follow the same path. Formally, at every stage p, $g^p(C) = g^p\left(C^{g^q}\right)$ and $\left\{\left(i^{p+1}(C), j^{p+1}(C)\right)\right\} = \left\{\left(i^{p+1}\left(C^{g^q}\right), j^{p+1}\left(C^{g^q}\right)\right)\right\}$ for all $q \in \{1, ..., \tau(C)\}$. Therefore, for all $i \in N$,

$$
\begin{aligned}
f^o_i(N_0, C) &= \sum_{p=1}^{|N|} c_{i^p j^p} \left[o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right) \right] \\
&= \sum_{p=1}^{|N|} \left(\left[o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right) \right] \sum_{q=1}^{\tau(C)} x^q c^{g^q}_{i^p j^p} \right) \\
&= \sum_{q=1}^{\tau(C)} x^q \left(\sum_{p=1}^{|N|} c^{g^q}_{i^p j^p} \left[o_i\left(S\left(P\left(g^{p-1}(C)\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C)\right), i\right)\right) \right] \right) \\
&= \sum_{q=1}^{\tau(C)} x^q \left(\sum_{p=1}^{|N|} c^{g^q}_{i^p j^p} \left[o_i\left(S\left(P\left(g^{p-1}(C^{g^q})\right), i\right)\right) - o_i\left(S\left(P\left(g^p(C^{g^q})\right), i\right)\right) \right] \right) \\
&= \sum_{q=1}^{\tau(C)} x^q f^o_i(N_0, C^{g^q})
\end{aligned}
$$

Now, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2: Let us denote

$$f_i'(N_0, C) = \sum_{S \subseteq N \backslash \{i\}} d_i^*(N \backslash S, o)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right]$$

We must prove that for any *mcstp* (N_o, C), $f'(N_o, C) = f^o(N_o, C)$.

By Lemma 8.2, there exists a family $\left\{\left(N_0, C^{g^q}\right)\right\}_{q=1}^{\tau(C)}$ of simple *mcstp* and a family $\{x^q\}_{q=1}^{\tau(C)}$ of non-negative real numbers such that $C = \sum\limits_{q=1}^{m(C)} x^q C^q$. By Lemma 8.4, $f^o(N_0, C) = \sum\limits_{q=1}^{\tau(C)} x^q f^o(N_0, C^{g^q})$. On the other hand, by Lemma 8.3, we have, $v_{C^*}(S) = \sum\limits_{q=1}^{\tau(C)} x^q v_{\left(C^{g^q}\right)^*}(S)$ for all $S \subseteq N$. Thus,

$$\begin{aligned} f_i'(N_0, C) &= \sum_{S \subseteq N \backslash \{i\}} d_i^*(N \backslash S, o)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right] \\ &= \sum_{S \subseteq N \backslash \{i\}} d_i^*(N \backslash S, o)\left[\sum_{q=1}^{\tau(C)} x^q \left(v_{\left(C^{g^q}\right)^*}(S \cup \{i\}) - v_{\left(C^{g^q}\right)^*}(S \cup \{i\})\right)\right] \\ &= \sum_{q=1}^{\tau(C)} x^q \left(\sum_{S \subseteq N \backslash \{i\}} d_i^*(N \backslash S, o)\left[\left(v_{\left(C^{g^q}\right)^*}(S \cup \{i\}) - v_{\left(C^{g^q}\right)^*}(S \cup \{i\})\right)\right]\right) \\ &= \sum_{q=1}^{\tau(C)} x^q f_i'\left(N_0, C^{g^q}\right) \end{aligned}$$

Hence it will be enough to prove that $f^o(N_0, C) = f'(N_0, C)$ when C is a simple problem.

C partition N_0 into $(S_0, S_1, ..., S_m)$ as follows. If $i, j \in S_k$, then $c_{ij} = 0$. If $i \in S_k$, $j \in S_l$, and $k \neq l$, then $c_{ij} = 1$. Without loss of generality we can assume that $0 \in S_0$. It is trivial to see that $f_i^o(N_0, C) = 0$ if $i \in S_0$ and $f_i^o(N_0, C) = o_i(S_k)$ if $i \in S_k$, $k \neq 0$.

We now compute $f_i'(N_0, C)$. Assume that $i \in S_0$. Since $v_{C^*}(S \cup \{i\}) = v_{C^*}(S)$ for all $S \subseteq N \backslash \{i\}$, we have that

$$f_i'(N_0, C) = 0 = f_i^o(N_0, C).$$

Assume that $i \in S_k$, $k \neq 0$. Since $v_{C^*}(S \cup \{i\}) = v_{C^*}(S)$ when $S \cap S_k \neq \emptyset$,

$$f_i'(N_0, C) = \sum_{S \cap S_k = \emptyset} d_i^*(N \backslash S, o)\left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S)\right]$$

Since $v_{C^*}(S \cup \{i\}) - v_{C^*}(S) = 1$ when $S \cap S_k = \emptyset$,

$$f_i'(N_0, C) = \sum_{S \cap S_k = \emptyset} d_i^*(N \backslash S, o) = \sum_{S_k \subseteq T \subseteq N} d_i^*(T, o) = o_i(S_k)$$

where the last equality follows from Equation 3.

8.3 Proof of Theorem 5.1

The following lemmata will be used in the proof of part (ii).

Lemma 8.5 *Let o^t be an obligation function for N, for all $t \in \{1, 2, \cdots, \kappa\}$. Suppose o is a convex combination of these obligation functions. That is, for all $i \in S$ and for all $S \subseteq N$, $o_i(S) = \sum_{t=1}^{\kappa} \alpha^t o_i^t(S)$, where $\sum_{t=1}^{\kappa} \alpha^t = 1$. Then $d(o) = \sum_{t=1}^{\kappa} \alpha^t d(o^t)$. Alternatively, if $d = \sum_{t=1}^{\kappa} \alpha^t d^t$, where d^t is an obligation function for N, for all $t = 1, 2, \cdots, \kappa$, then $o(d) = \sum_{t=1}^{\kappa} \alpha^t o(d^t)$.*

Proof of Lemma 8.5: One can easily check that indeed o is a valid obligation function for N, that is it satisfies o-i and o-ii. Thus $d(o)$ is also well defined. We will use induction to show that for $i \in S$ and for all $S \subseteq \partial(o)^2$, $d_i(S, o) = \sum_{t=1}^{\kappa} \alpha^t d_i(S, o^t)$. This is easy to show when $S = \partial(o)$, because $d_i(\partial(o), o) = o_i(\partial(o)) = \sum_{t=1}^{\kappa} \alpha^t o_i^t(\partial(o)) = \sum_{t=1}^{\kappa} \alpha^t d_i(\partial(o), o^t)$. Suppose we have proved this result for all S, such that $|\partial(o) \setminus S| < \kappa$. We will prove the same when $|\partial(o) \setminus S| = \kappa$. Without loss of generality assume that $k \notin S$. Since $o = \sum_{t=1}^{\kappa} \alpha^t o^t$, we must have $o^{-k} = \sum_{t=1}^{\kappa} \alpha^t \left(o^t\right)^{-k}$. Thus

$$\begin{aligned}
d_i(S, o) &= d_i(S, o^{-k}) - d_i(S \cup \{k\}, o) \quad \text{(by Equation 2)} \\
&= \sum_{t=1}^{\kappa} \alpha^t d_i(S, \left(o^t\right)^{-k}) - \sum_{t=1}^{\kappa} \alpha^t d_i(S \cup \{k\}, o^t) \quad \text{(by induction)} \\
&= \sum_{t=1}^{\kappa} \alpha^t \left[d_i(S, \left(o^t\right)^{-k}) - d_i(S \cup \{k\}, o^t)\right] \\
&= \sum_{t=1}^{\kappa} \alpha^t d_i(S, o^t) \quad \text{(by Equation 2)}
\end{aligned}$$

Alternatively, suppose $d = \sum_{t=1}^{\kappa} \alpha^t d^t$, That is for all $S \subseteq \partial(d)$ and for all $i \in S$, $d_i(S) = \sum_{t=1}^{\kappa} \alpha^t d^t(S)$. By Equation 3,

$$o_i(S, d) = \sum_{S \subseteq T \subseteq \partial(d)} d_i(T) = \sum_{S \subseteq T \subseteq \partial(d)} \sum_{t=1}^{\kappa} \alpha^t d_i^t(T) = \sum_{t=1}^{\kappa} \alpha^t \sum_{S \subseteq T \subseteq \partial(d)} d_i^t(T) = \sum_{t=1}^{\kappa} \alpha^t o_i(S, d^t)$$

This completes the proof of Lemma 8.5.

[2] $\partial(o) = N$

Lemma 8.6 *Suppose there are two sets of nonnegative real numbers* $\{\theta_1, \theta_2, \cdots, \theta_s\}$ *and* $\{\theta^1, \theta^2, \cdots, \theta^t\}$ *such that* $\sum_{i=1}^{s} \theta_i = \sum_{j=1}^{t} \theta^j$. *Then for all* $i \leq s$ *and for all* $j \leq t$, *we can find nonnegative real numbers* θ_i^j *which satisfy the following conditions.* (a) *For all* $j \leq t$, $\sum_{i=1}^{s} \theta_i^j = \theta^j$. (b) *For all* $i \leq s$, $\sum_{j=1}^{t} \theta_i^j = \theta_i$.

Proof of Lemma 8.6: We will prove this result by induction on s. If $s = 1$, we can choose $\theta_1^j = \theta^j$ for all $j \leq t$. Thus condition (a) is trivially satisfied. Since $\theta^j \geq 0$, we have $\theta_1^j \geq 0$. Condition (b) is satisfied because $\theta_1 = \sum_{j=1}^{t} \theta^j = \sum_{j=1}^{t} \theta_1^j$. Let us now assume that our hypothesis is true for all $s < \kappa$. We will prove that the same for $s = \kappa$. Let us first divide θ_s into $\{\theta_s^j\}_{j=1}^t$ as follows. Let $\theta_s^1 = \min(\theta_s, \theta^1)$. If $\theta_s < \theta^1$, that is $\theta_s^1 = \theta_s$, then there is nothing to be distributed among $\{\theta_s^j\}_{j=2}^t$ and hence $\theta_s^j = 0$ for all $j = 2, 3, \cdots, t$. Otherwise $\theta_s^1 = \theta^1$ and we distribute the residual of θ_s, that is $(\theta_s - \theta^1)$ among $\{\theta_s^j\}_{j=2}^t$. We follow this procedure repeatedly and at each step set $\theta_s^j = \min($residual of $\theta_s, \theta^j)$. Whenever the residual is 0 all the following θ_s^j are set to 0. Formally, for all $j \leq t$

$$\theta_s^j = \min\left(\max\left([\theta_s - \sum_{k=1}^{j-1} \theta^k], 0\right), \theta^j\right)$$

Since $\sum_{i=1}^{s} \theta_i = \sum_{j=1}^{t} \theta^j$ and $\{\theta_i\}_{i=1}^s$ are nonnegative, we must have $\theta_s \leq \sum_{j=1}^{t} \theta^j$, that is there will be no residuals left after the $t-$th step. In other words, $\sum_{j=1}^{t} \theta_s^j = \theta_s$. Also note that by construction, $\theta_s^j \geq 0$ for all $j \leq t$. This reduces the construction to a smaller problem where we can use induction. Formally, the residual problem is represented by $\{\theta_1, \theta_2, \cdots, \theta_{s-1}\}$ and $\{\bar{\theta}^1, \bar{\theta}^2, \cdots, \bar{\theta}^t\}$, where for all $j \leq t$, $\bar{\theta}^j = (\theta^j - \theta_s^j)$. Note that $\sum_{i=1}^{(s-1)} \theta_i = \sum_{i=1}^{s} \theta_i - \theta_s = \sum_{j=1}^{t} \theta^j - \sum_{j=1}^{t} \theta_s^j = \sum_{j=1}^{t} \bar{\theta}^j$. By induction hypothesis, there exists nonnegative θ_i^j for all $i \leq (s-1)$ and $j \leq t$ such that (a) and (b) are satisfied for the residual problem. Now it is easy to check that these $\{\theta_i^j\}_{i \leq (s-1),\, j \leq t}$ along with $\{\theta_s^j\}_{j \leq t}$ constitute a possible construction for the original problem.

(a) For all $j \leq t$, $\sum_{i=1}^{s} \theta_i^j = \theta_s^j + \sum_{i=1}^{(s-1)} \theta_i^j = \theta_s^j + \bar{\theta}^j = \theta^j$. Note that $\sum_{i=1}^{(s-1)} \theta_i^j = \bar{\theta}^j$ is ensured by the induction hypothesis.

(b) Similarly by induction hypothesis $\sum_{j=1}^{t} \theta_i^j = \theta_i$ for all $i < s$. It has been already shown that $\sum_{j=1}^{t} \theta_s^j = \theta_s$.

Lemma 8.7 *Let $\pi \in \Pi_N$. An obligation function o^π is defined as follows. For each $S \subset N$ and $i \in S$,*

$$o_i^\pi(S) = \begin{cases} 1 & \text{if } \pi(i) = \min\limits_{j \in S} \{\pi(j)\} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Then,

$$d_i(S, o^\pi) = \begin{cases} 1 & \text{if } S = \{j | \pi(j) \geq |N \setminus S|\} \text{ and } \pi(i) = \min_{j \in S} \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

Proof of Lemma 8.7: Once again we use induction on $|\partial(o^\pi) \setminus S|$. This is easily done when $|\partial(o^\pi) \setminus S| = 0$, because $d_i(\partial(o^\pi), o^\pi) = o_i^\pi(\partial(o^\pi))$, which is 1 for i such that $\pi(i) = \min_{j \in \partial(o^\pi)} \pi(j)$ and 0 otherwise. Suppose our assertion is true for all S such that $|\partial(o^\pi) \setminus S| < \kappa$. We will show the same when $|\partial(o^\pi) \setminus S| = \kappa$. Choose any $k \notin S$. Let us define an order π^{-k} on $N \setminus \{k\}$ as follows. For all i, $j \in N \setminus \{k\}$, $\pi^{-k}(i) < \pi^{-k}(j) \Leftrightarrow \pi(i) < \pi(j)$. One can easily check that $o^{(\pi^{-k})} = (o^\pi)^{-k}$. Therefore

$$\begin{aligned} d_i(S, o^\pi) &= \left[d_i(S, (o^\pi)^{-k}) - d_i(S \cup \{k\}, o^\pi) \right] \quad \text{(by Equation 3)} \\ &= \left[d_i(S, o^{(\pi^{-k})}) - d_i(S \cup \{k\}, o^\pi) \right] \end{aligned}$$

Now, if $S = \{j | \pi(j) \geq |\partial(o^\pi) \setminus S|\}$ then $\pi(k) < \pi(j)$ for all $j \in S$. Hence $S = \left\{ j | \pi^{-k}(j) \geq |\partial\left(o^{(\pi^{-k})}\right) \setminus S| \right\}$ and $\pi(k) = \min_{j \in S \cup \{k\}} \pi(j)$. Thus by induction hypothesis,

$$d_i(S, o^\pi) = \left[d_i(S, o^{(\pi^{-k})}) - d_i(S \cup \{k\}, o^\pi) \right] = \begin{cases} (1-0) = 1 & \text{if } \pi(i) = \min_{j \in S} \pi(j) \\ (0-0) = 0 & \text{otherwise} \end{cases}$$

For all other S, if possible let us pick an k, such that $\pi(k) < \pi(j)$ for all $j \in S$. Once again $\pi(k) = \min_{j \in S \cup \{k\}} \pi(j)$, but $S \neq \left\{ j | \pi^{-k}(j) \geq |\partial\left(o^{(\pi^{-k})}\right) \setminus S| \right\}$. By induction, for all $i \in S$,

$$d_i(S, o^\pi) = \left[d_i(S, o^{(\pi^{-k})}) - d_i(S \cup \{k\}, o^\pi) \right] = 0 - 0 = 0$$

Finally, all S such that $S = \partial(o) \setminus \{k\}$, where $k \neq \min_{j \in \partial(o)} \pi(j)$ are not covered by the previous cases. For these sets, $S = \partial\left(o^{(\pi^{-k})}\right)$ and $S \cup \{k\} = \partial(o)$. Thus $S = \left\{ j | \pi^{-k}(j) \geq |\partial\left(o^{(\pi^{-k})}\right) \setminus S| \right\}$ and $S \cup \{k\} = \{j | \pi(j) \geq |\partial(o^\pi) \setminus (S \cup \{k\})|\}$. Moreover, since $k \neq \min_{j \in \partial(o)} \pi(j)$, we have $\pi(i) = \min_{j \in S} \pi(j) \Leftrightarrow \pi(i) = \min_{j \in S \cup \{k\}} \pi(j)$. Therefore by induction

$$d_i(S, o^\pi) = \left[d_i(S, o^{(\pi^{-k})}) - d_i(S \cup \{k\}, o^\pi) \right] = \begin{cases} (1-1) = 0 & \text{if } \pi(i) = \min_{j \in S} \pi(j) \\ (0-0) = 0 & \text{otherwise} \end{cases}$$

This completes the proof of Lemma 8.7.

Lemma 8.8 *Let $\bar{\pi} \in \Pi_N$. An obligation function $o^{\bar{\pi}}$ is defined as in Equation 7. Then $f^{o^{\bar{\pi}}} \in W$.*

Proof of Lemma 8.8: From Equation 1, we know that if we can find a weight system $(\hat{w}_\pi)_{\pi \in \Pi_N} \in \Delta(\Pi_N)$ such that

$$f_i^{o^{\bar{\pi}}}(N_0, C) = \sum_{S \subseteq N \setminus \{i\}} \left(\sum_{\{\pi | Pre(i,\pi)=S\}} \hat{w}_\pi \right) \left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S) \right]$$

for all $i \in N$ and for all *mcstp* (N_0, C) then we are done. By Equation 4, for all $i \in N$ and for all *mcstp* (N_0, C)

$$f_i^{o^{\bar{\pi}}}(N_0, C) = \sum_{S \subseteq N \setminus \{i\}} d_i\left(N \setminus S, o^{\bar{\pi}}\right) \left[v_{C^*}(S \cup \{i\}) - v_{C^*}(S) \right]$$

Thus all we need to find is a weight system $(\hat{w}_\pi)_{\pi \in \Pi_N} \in \Delta(\Pi_N)$, such that for all $i \in S$ and for all $S \subseteq N$,

$$d_i\left(S, o^{\bar{\pi}}\right) = \sum_{\{\pi | Pre(i,\pi)=N \setminus S\}} \hat{w}_\pi$$

Let us consider the following weight system $(\hat{w}_\pi)_{\pi \in \Pi_N}$,

$$\hat{w}_\pi = \begin{cases} 1 & \text{if } \pi = \bar{\pi} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, for all $i \in S$ and for all $S \subseteq N$

$$\begin{aligned} \sum_{\{\pi | Pre(i,\pi)=N \setminus S\}} \hat{w}_\pi &= \begin{cases} 1 & \text{if } S = \{j | \bar{\pi}(j) \geq |N \setminus S|\} \text{ and } \bar{\pi}(i) = \min_{j \in S} \bar{\pi}(j) \\ 0 & \text{otherwise} \end{cases} \\ &= d_i\left(S, o^{\bar{\pi}}\right) \qquad \text{(by Lemma 8.7)} \end{aligned}$$

Lemma 8.9 *Let $w \in \Delta(\Pi_N)$ and $o^w = \sum_{\pi \in \Pi_N} w_\pi o^\pi$ and o^π is given by Equation 7. Then $f^{o^w} \in W$.*

Proof of Lemma 8.9: From the definition of the obligation rules using Kruskal's algorithm, for all $i \in N$ and for all *mcstp* (N_0, C)

$$\begin{aligned} f_i^{o^w}(N_0, C) &= \sum_{p=1}^{|N|} c_{i^p j^p} \left[o_i^w\left(S\left(P\left(g^{p-1}\right), i\right)\right) - o_i^w\left(S\left(P\left(g^p\right), i\right)\right) \right] \\ &= \sum_{p=1}^{|N|} c_{i^p j^p} \left[\sum_{\pi \in \Pi_N} w_\pi o_i^\pi\left(S\left(P\left(g^{p-1}\right), i\right)\right) - \sum_{\pi \in \Pi_N} w_\pi o_i^\pi\left(S\left(P\left(g^p\right), i\right)\right) \right] \\ &= \sum_{\pi \in \Pi_N} w_\pi \left[\sum_{p=1}^{|N|} c_{i^p j^p} \left[o_i^\pi\left(S\left(P\left(g^{p-1}\right), i\right)\right) - o_i^\pi\left(S\left(P\left(g^p\right), i\right)\right) \right] \right] \\ &= \sum_{\pi \in \Pi_N} w_\pi f_i^{o^\pi}(N_0, C) \end{aligned}$$

The result follows immediately from Lemma 8.8 and the fact that W is a convex set.

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1: This proof has three steps.

(i) $W \subseteq O$.

(ii) Let o be an obligation function. $f^o \in W \Leftrightarrow d_i(S, o) \geq 0$ for all $i \in S$ and for all $S \subseteq N$.

(iii) If $|N| > 3$, then $W \subset O$.

Proof of (i): Suppose $f^w \in W$. Let (N_0, C) be a *mcstp*. From Equation 1, we know that

$$f_i^w(N_0, C) = \sum_{S \subseteq N \backslash \{i\}} \left(\sum_{\{\pi | Pre(i,\pi) = S\}} w_\pi \right) [v_{C^*}(S \cup \{i\}) - v_{C^*}(S)]$$

Let us define for all $S \subseteq N$ and for all $i \in N \setminus S$,

$$\hat{d}_i(N \backslash S, o) = \sum_{\{\pi | Pre(i,\pi) = S\}} w_\pi$$

Thus

$$f_i^w(N_0, C) = \sum_{S \subseteq N \backslash \{i\}} \hat{d}_i(N \backslash S, o) [v_{C^*}(S \cup \{i\}) - v_{C^*}(S)]$$

To complete the proof, we need to show that $\hat{d}$ is an obligation rule, that is, it satisfies d-i and d-ii.

Checking d-i:

$$\begin{aligned}
\sum_{i \in N} \hat{d}_i(N) &= \sum_{i \in N} \sum_{\{\pi | Pre(i,\pi) = \emptyset\}} w_\pi \\
&= \sum_{i \in N} \sum_{\{\pi | \pi(i) = 1\}} w_\pi \\
&= \sum_{\pi \in \Pi_N} w_\pi = 1
\end{aligned}$$

Also for all $S \subset N$,

$$\begin{aligned}
&\sum_{j \in N \backslash S} \hat{d}_j(S \cup \{j\}) \\
&= \sum_{j \in N \backslash S} \left[\sum_{\{\pi | Pre(j,\pi) = N \backslash [S \cup \{j\}]\}} w_\pi \right]
\end{aligned}$$

$$= \sum_{j \in N \setminus S} \left[\sum_{i \in S} \left[\sum_{\substack{Pre(j,\pi) = N \setminus [S \cup \{j\}] \\ \pi(i) = \pi(j)+1}} w_\pi \right] \right]$$

$$= \sum_{i \in S} \left[\sum_{j \in N \setminus S} \left[\sum_{\substack{Pre(i,\pi) = N \setminus S \\ \pi(j) = \pi(i)-1}} w_\pi \right] \right]$$

$$= \sum_{i \in S} \left[\sum_{\{\pi | Pre(i,\pi) = N \setminus S\}} w_\pi \right] = \sum_{i \in S} \hat{d}_i(S)$$

Checking d-ii: This follows trivially from the fact that $w_\pi \geq 0$ for all $\pi \in \Pi_N$.

Proof of (ii):

Proof of " $\Rightarrow$ ": This follows immediately from part (i). For all $S \subseteq N$ and for all $i \in S$, $d_i(S, o) = \sum_{\{\pi | Pre(i,\pi) = N \setminus S\}} w_\pi \geq 0$ because $w_\pi \geq 0$ for all $\pi \in \Pi_N$.

Proof of " $\Leftarrow$ ": Let o be an obligation function on N such that

$$d_i(S, o) \geq 0 \text{ for all } i \in S \text{ and for all } S \subseteq N \tag{8}$$

If we can show that $o = o^w$ for some $w \in \Delta(\Pi_N)$, where $o^w = \sum_{\pi \in \Pi_N} w_\pi o^\pi$ and o^π is given by Equation 7, then by Lemma 8.9, $f^o \in W$. In the rest of this proof we use induction on $|N|$ to show, $o = \sum_{\pi \in \Pi_N} w_\pi o^\pi$ for some $w \in \Delta(\Pi_N)$.

Suppose $|N| = 2$. Without loss of generality we can assume $N = \{1, 2\}$. From d-i we know that $d_1(\{1,2\}, o) + d_2(\{1,2\}, o) = 1$. Moreover $d_1(\{1\}, o) = d_2(\{1,2\}, o)$ and $d_2(\{2\}, o) = d_1(\{1,2\}, o)$. By assumption 8, $d_1(\{1\}, o)$, $d_2(\{2\}, o)$, $d_1(\{1,2\}, o)$ and $d_2(\{1,2\}, o)$ are all non negative[3]. It can be easily checked that o can be written as follows,

$$o = d_1(\{1,2\}, o) \cdot o^{\pi_1} + d_2(\{1,2\}, o) \cdot o^{\pi_2}$$

where π_1 and π_2 denote the natural order and it's reverse respectively. Now, let us assume that our assertion is true for all N such that $|N| < \kappa$. We show that the same is true when $|N| = \kappa$. Once again we use a chain of claims to establish this result. The scheme of the proof is as follows.

Claim 7: There exist obligation functions $\{d^k\}_{k \in N}$ on N such that $d(o) = \sum_{k \in N} d_k(N, o) \cdot d^k$, where for all $k \in N$, d^k satisfies the following conditions.

[3] By d-ii these are anyway non negative, afact which we explore in corollary

(a) For all $S \subseteq N$ and for all $i \in S$, $d_i^k(S) \geq 0$.

(b) $d_i^k(N) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$

(c) If $k \in S \subset N$, then for all $i \in S$, $d_i^k(S) = 0$

Claim 8: Suppose $(d^k)^{-k}$ represents the restriction of d^k on $N \setminus \{k\}$. That is for all $S \subseteq N \setminus \{k\}$ and for all $i \in S$, $(d^k)_i^{-k}(S) = d_i^k(S)$. Then $(d^k)^{-k}$ is an obligation function on $N \setminus \{k\}$. In fact $o\left((d^k)^{-k}\right) = [o(d^k)]^{-k}$.

Supposing claim 7 and claim 8 are true, we can proceed as follows. Note that by claim 7, $\left(d^k\right)_i^{-k}(S) = d_i^k(S) \geq 0$. Since $(d^k)^{-k}$ is an obligation function on $N \setminus \{k\}$, by induction hypothesis and claim 8, we have

$$[o(d^k)]^{-k} = o\left((d^k)^{-k}\right) = \sum_{\pi^{-k} \in \Pi_{N \setminus \{k\}}} w_{\pi^{-k}} \cdot o^{(\pi^{-k})} \tag{9}$$

where $\Pi_{N \setminus \{k\}}$ is the set of all orders on $N \setminus \{k\}$ and $w \in \Delta(\Pi_{N \setminus \{k\}})$. Now, we can use π^{-k} to define a new order (k, π^{-k}) on N. (k, π^{-k}) starts with k and then follows exactly the same sequence as in π^{-k}. For all $k \in N$ and for all $\pi^{-k} \in \Pi_{N \setminus \{k\}}$ we can define obligation functions $o^{(k, \pi^{-k})}$ by Equation 7.

Claim 9: $o(d^k) = \sum_{\pi^{-k} \in \Pi_{N \setminus \{k\}}} w_{\pi^{-k}} \cdot o^{(k, \pi^{-k})}$

By claim 7, we have $d(o) = \sum_{k \in N} d_k(N, o) \cdot d^k$. Using Lemma 8.5,

$$\begin{aligned}
o &= \sum_{k \in N} d_k(N, o) \cdot o(d^k) \\
&= \sum_{k \in N} d_k(N, o) \left[\sum_{\pi^{-k} \in \Pi_{N \setminus \{k\}}} w_{\pi^{-k}} \cdot o^{(k, \pi^{-k})} \right] \quad \text{(By claim 9)} \\
&= \sum_{k \in N} \sum_{\pi^{-k} \in \Pi_{N \setminus \{k\}}} [d_k(N, o) \cdot w_{\pi^{-k}}] o^{(k, \pi^{-k})} \\
&= \sum_{\pi \in \Pi_N} [d_k(N, o) \cdot w_{\pi^{-k}}] o^{\pi}
\end{aligned}$$

The last inequality follows from the fact $\Pi_N = \cup_{k \in N} \{\pi | \pi(k) = 1\} = \cup_{k \in N} \{(k, \pi^{-k}) | \pi^{-k} \in \Pi_{N \setminus \{k\}}\}$.

For all $\pi \in \Pi_N$, let us denote the coefficient of o^{π} by $\bar{w}_{\pi}$. That is $\bar{w}_{\pi} = d_k(N, o) \cdot w_{\pi^{-k}}$. Since, $d_k(N, o) \geq 0$ and $w_{\pi^{-k}} \geq 0$, we have $\bar{w}_{\pi} \geq 0$. Moreover

$$\begin{aligned}
\sum_{\pi \in \Pi_N} \bar{w}_{\pi} &= \sum_{\pi \in \Pi_N} [d_k(N, o) \cdot w_{\pi^{-k}}] \\
&= \sum_{k \in N} \sum_{\pi^{-k} \in \Pi_{N \setminus \{k\}}} [d_k(N, o) \cdot w_{\pi^{-k}}]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in N} d_k(N,o) \sum_{\pi^{-k}\in\Pi_{N\setminus\{k\}}} w_{\pi^{-k}} \\
&= \sum_{k\in N} d_k(N,o) \quad (\text{as } w\in\Delta(\Pi_{N\setminus\{k\}})) \\
&= 1 \quad (\text{by d-i})
\end{aligned}
$$

Thus indeed $\bar{w} \in \Delta(\Pi_N)$ and $o = \sum_{\pi\in\Pi_N} \bar{w}_\pi o^\pi$. To complete the proof of part (iii), now we get back to the proves of claim 7, 8 and 9.

Proof of claim 7: Without loss of generality we can assume that $d_k(N,o) > 0$ for all $k \in N$ (otherwise, if there exist an l such that $d_l(N,o) = 0$, then choose any obligation function d^l satisfying (a), (b) and (c) and apply rest of the proof on $N \setminus \{l\}$). We need to show that there exist obligation functions $\{d^k\}_{k\in N}$ on N such that $d(o) = \sum_{k\in N} d_k(N,o)\cdot d^k$, where for all $k\in N$, d^k satisfies properties (a), (b) and (c). Note that if we can construct such $\{d^k\}_{k\in N}$, then each d^k will trivially satisfy the property d-ii of obligation rules because of (a)[4]. Moreover property (b) will ensure $\sum_{i\in N} d_i^k(N) = 1$. So while constructing $\{d^k\}_{k\in N}$, apart from (a), (b) and (c), we need to ensure the following properties

- (d) A part of property d-i, which is, $\sum_{i\in S} d_i^k(S) = \sum_{j\notin S} d_j^k(S\cup j)$ for all $S \subset N$ for all $k\in N$.
- (e) $\sum_{k\in N} [d_k(N,o)\cdot d_i^k(S)] = d_i(S,o)$, for all $i\in S$ and for all $S\subseteq N$.

We will use induction on $|S|$. If $S = N$, then the possibility of such a construction is easy to check. Take any $i \in N$. By (b), $\sum_{k\in N}[d_k(N,o)\cdot d_i^k(S)] = d_i(N,o)\cdot d_i^i(S) = d_i(N,o)$. Suppose we have been able to construct $\{d^k(S)\}_{k\in N}$ for all $|S| > \kappa$, satisfying the properties listed above. We will now construct the same for $S\subset N$ such that $|S| = \kappa$. Let us fix such an S.

To satisfy (c), we set $d_i^k(S) = 0$ for all $i\in S$ and $k\in S$. This will change requirements (d) and (e) as follows,

- (d') $\sum_{i\in S} d_i^k(S) = \sum_{j\notin S} d_j^k(S\cup j)$ for all $k \in N\setminus S$. For $k\in S$, this is trivially true, as by construction $\sum_{i\in S} d_i^k(S) = 0 = \sum_{j\notin S} d_j^k(S\cup j)$. The second inequality follows from the induction step, property (c) and the fact that $k\in S \Rightarrow k\in S\cup\{j\}$ for all $j\notin S$.

[4]Property (a) implies that $d_i^k(N) \geq 0$ for all $i\in N$. Similarly for all $S\subset N$, for all $l\in N\setminus S$ and for all $i\in S$, $\sum_{S\subseteq T\subseteq N\setminus\{l\}} d_i^k(T) \geq 0$.

(e') $d_i(S,o) = \sum\limits_{k\in N}[d_k(N,o)\cdot d_i^k(S)] = \sum\limits_{k\in N\setminus S}[d_k(N,o)\cdot d_i^k(S)] + \sum\limits_{k\in S}[d_k(N,o)\cdot d_i^k(S)] = \sum\limits_{k\in N\setminus S}[d_k(N,o)\cdot d_i^k(S)]$.

Now we construct the remaining elements, that is $\{d_i^k(S)\}_{i\in S,k\in N\setminus S}$. Let $\theta^k = d_k(N,o)\cdot\sum\limits_{j\notin S} d_j^k(S\cup j)$ for all $k\in N\setminus S$. Let $\theta_i = d_i(S,o)$ for all $i\in S$. Note that, by Equation 8 $\{\theta_i\}_{i\in S}$ are nonnegative numbers and by induction hypothesis $\{\theta^k\}_{k\in N\setminus S}$ are also nonnegative. Moreover,

$$\begin{aligned}
\sum_{k\in N\setminus S}\theta^k &= \sum_{k\in N\setminus S} d_k(N,o)\left[\sum_{j\notin S} d_j^k(S\cup j)\right]\\
&= \sum_{j\notin S}\sum_{k\in N\setminus S} d_k(N,o)\cdot d_j^k(S\cup j)\\
&= \sum_{j\notin S}\left[\sum_{k\in N\setminus S} d_k(N,o)\cdot d_j^k(S\cup j) + \sum_{k\in S} d_k(N,o)\cdot d_j^k(S\cup j)\right]
\end{aligned}$$

The last equality follows from the induction hypothesis and property (c), as $k\in S$ implies $d_j^k(S\cup j)=0$ for all $j\notin S$. Thus

$$\begin{aligned}
\sum_{k\in N\setminus S}\theta^k &= \sum_{j\notin S}\left[\sum_{k\in N\setminus S} d_k(N,o)\cdot d_j^k(S\cup j) + \sum_{k\in S} d_k(N,o)\cdot d_j^k(S\cup j)\right]\\
&= \sum_{j\notin S}\left[\sum_{k\in N} d_k(N,o)\cdot d_j^k(S\cup j)\right]\\
&= \sum_{j\notin S} d_j(S\cup j,o) \quad \text{(by induction hypothesis and } (e)\text{)}\\
&= \sum_{i\in S} d_i(S,o) \quad \text{(by property d-i of the obligation function } d(o)\text{)}\\
&= \sum_{i\in S}\theta_i
\end{aligned}$$

Applying Lemma 8.6 on $\{\theta_i\}_{i\in S}$, $\{\theta^k\}_{k\in N\setminus S}$, we can find nonnegative real numbers $\{\theta_i^k\}_{i\in S,k\in N\setminus S}$ such that $\sum\limits_{i\in S}\theta_i^k = \theta^k$ for all $k\in N\setminus S$ and $\sum\limits_{k\in N\setminus S}\theta_i^k = \theta_i$ for all $i\in S$. For all $i\in S$ and $k\in N\setminus S$, let $d_i^k(S) = \frac{\theta_i^k}{d_k(N,o)}$. This completes the description of $\{d^k(S)\}_{k\in N}$. By construction, they satisfy properties (a), (b) and (c). To complete the induction step, we now show that (d') and (e') are also satisfied.

Checking (d'): For $k \in N\backslash S$; we have $\sum\limits_{i\in S} d_i^k(S) = \frac{1}{d_k(N,o)} \sum\limits_{i\in S} \theta_i^k = \frac{1}{d_k(N,o)}\theta^k = \frac{1}{d_k(N,o)} \left[d_k(N,o) \cdot \sum\limits_{j\notin S} d_j^k(S\cup j) \right] = \sum\limits_{j\notin S} d_j^k(S\cup j)$.

Checking (e'): For all $i \in S$; we have $\sum\limits_{k\in N\backslash S} [d_k(N,o)\cdot d_i^k(S)] = \sum\limits_{k\in N\backslash S} [d_k(N,o)\cdot d_i^k(S)] = \sum\limits_{k\in N\backslash S} \left[d_k(N,o) \cdot \frac{\theta_i^k}{d_k(N,o)} \right] = \sum\limits_{k\in N\backslash S} \theta_i^k = \theta_i = d_i(S,o)$.

Proof of claim 8: We will show that $o\left((d^k)^{-k}\right) = [o(d^k)]^{-k}$, that is for all $S \subseteq N \setminus \{k\}$, for all $i \in S$, $o_i\left(S, (d^k)^{-k}\right) = o_i\left(S, d^k\right)$. By Equation 3, $o_i\left(S, d^k\right) = \sum\limits_{S\subseteq T\subseteq N} d_i^k(T) = \sum\limits_{S\subseteq T\subseteq N\backslash\{k\}} d_i^k(T) + \sum\limits_{S\subseteq T\subset N\backslash\{k\}} d_i^k(T\cup\{k\}) + d_i^k(N)$. From property ($c$), we have $d_i^k(T\cup\{k\}) = 0$ for all $T \subset N\backslash\{k\}$ and from property (b), $d_i^k(N) = 0$ because $i \neq k$. Therefore $o_i\left(S, d^k\right) = \sum\limits_{S\subseteq T\subseteq N\backslash\{k\}} d_i^k(T) = \sum\limits_{S\subseteq T\subseteq N\backslash\{k\}} (d^k)_i^{-k}(T) = o_i\left(S, (d^k)^{-k}\right)$. Thus $(d^k)^{-k}$ is an obligation rule on $N \setminus \{k\}$.

Proof of claim 9: We need to prove $o(d^k) = \sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o^{(k,\pi^{-k})}$, that is for all $S \subseteq N$ and for all $i \in S$, $o_i(S, d^k) = \sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{(k,\pi^{-k})}(S)$. Let us remind the readers that $o^{(k,\pi^{-k})}$ is defined by Equation 7 and d^k satisfies the properties of claim 7. $\hat{\pi} = (k, \pi^{-k})$ represents an order on N, where $\hat{\pi}(k) = 1$.

Suppose $S \subseteq N\backslash\{k\}$. From Equation 7 it is immediate that $(o^{\hat{\pi}})^{-k} = o^{(\pi^{-k})}$. Then for all $i \in S$, $\sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{\hat{\pi}}(S) = \sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{(\pi^{-k})}(S)$. Thus by Equation 9, $\sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{\hat{\pi}}(S) = o_i(S, d^k)$.

Otherwise if $k \in S$, then for all $i \in S$ and $i \neq k$, we have $o_i^{\hat{\pi}}(S) = 0$. Hence $\sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{\hat{\pi}}(S) = \sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot 0 = 0$. Since $o_k^{\hat{\pi}}(S) = 1$ and $w \in \Delta(\Pi_{N\backslash\{k\}})$, we have $\sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_k^{\hat{\pi}}(S) = \sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot 1 = 1$. On the other hand by property (c), $o(S, d^k) = \sum\limits_{S\subseteq T\subseteq N} d^k(T) = d^k(N)$. Thus by property ($b$), $o_i(S, d^k) = 0$ for all $i \in S$, $i \neq k$ and $o_k(S, d^k) = 1$. Hence $\sum\limits_{\pi^{-k}\in\Pi_{N\backslash\{k\}}} w_{\pi^{-k}} \cdot o_i^{\hat{\pi}}(S) = o_i(S, d^k)$ for all $i \in S$.

Proof of (*iii*): It will be enough to construct an obligation function $\hat{o}$ which violates the condition $d_i(S, o) \geq 0$ for some $i \in S$ and $S \subseteq N$. Note that such an obligation function has already been described in example 4.3 when $|N| = 4$. We will extend that example here for arbitrary N.

Let $N = \{1, 2, \cdots, n\}$. An obligation function $\hat{o}$ is described as follows. For all $S \subseteq N$ and for all $i \in S$, $\hat{o}_i(S) = \frac{1}{|S|}$ except when $S = \{1, 2, \cdots (n-1)\}$.

$$\hat{o}_i(\{1, 2, \cdots, (n-1)\}) = \begin{cases} \frac{1}{(n-2)} & \text{if } i = 1 \\ \frac{(n-3)}{(n-2)^2} & \text{otherwise} \end{cases}$$

It is easy to check that $\hat{o}$ satisfy o-i. Let us verify that $\hat{o}$ satisfies o-ii. We have to show for all $S \subset T$ and for $i \in S$, $\hat{o}_i(S) \geq \hat{o}_i(T)$. First consider all S, T such that $S,T \neq \{1, 2, \cdots (n-1)\}$. Then $\hat{o}_i(S) = \frac{1}{|S|} > \frac{1}{|T|} = \hat{o}_i(T)$ for all $i \in S$. Next, suppose $S = \{1, 2, \cdots (n-1)\}$. There is only one superset of $\{1, 2, \cdots (n-1)\}$, that is $T = N$. Thus $\hat{o}_1(S) = \frac{1}{(n-2)} > \frac{1}{n} = \hat{o}_1(T)$. For all $i \in S \setminus \{1\}$, we have, $\hat{o}_i(S) = \frac{(n-3)}{(n-2)^2} \geq \frac{1}{n} = \hat{o}_i(T)$, because $|N| = n \geq 4$. Finally if $T = \{1, 2, \cdots (n-1)\}$, then $|S| \leq (n-2)$. Therefore, in case $1 \in S$, we have $\hat{o}_1(S) = \frac{1}{|S|} \geq \frac{1}{(n-2)} = \hat{o}_1(T)$. For all $i \in S \setminus \{1\}$, $\hat{o}_i(S) = \frac{1}{|S|} \geq \frac{1}{(n-2)} > \frac{(n-3)}{(n-2)^2} = \hat{o}_i(T)$. Hence $\hat{o}$ satisfies o-ii and it is indeed an obligation function. However note that

$$\begin{aligned}
& d_1^*(\{1, 2, \cdots, (n-2)\}, \hat{o}) \\
&= d_1^*(\{1, 2, \cdots, (n-2)\}, \hat{o}^{-(n-1))}) - d_1^*(\{1, 2, \cdots, (n-1)\}, \hat{o}) \\
&= [d_1^*\left(\{1, 2, \cdots, (n-2)\}, (\hat{o}^{-(n-1)})^{-n}\right) - d_1^*(\{1, 2, \cdots, (n-2), n\}, \hat{o}^{-(n-1)})] \\
&\qquad - [d_1^*(\{1, 2, \cdots, (n-1)\}, \hat{o}^{-n}) - d_1^*(N, \hat{o})] \\
&= [\hat{o}_1(\{1, 2, \cdots, (n-2)\}) - \hat{o}_1(\{1, 2, \cdots, (n-2), n\})] \\
&\qquad - [\hat{o}_1(\{1, 2, \cdots, (n-1)\}) - \hat{o}_1(N)] \\
&= \left(\frac{1}{(n-2)} - \frac{1}{(n-1)} - \frac{1}{(n-2)} + \frac{1}{n}\right) = -\frac{1}{n(n-1)} < 0
\end{aligned}$$

Hence $f^{\hat{o}} \notin W$. This completes the proof.

8.4 Proof of Corollary 5.2

Proof of " $\Rightarrow$ ": Let $f^w \in W$. For each $\pi \in \Pi_N$, let o^π be the obligation function defined as in Claim 2 of the proof of Theorem 3.

Because of the proof of STEP 1 of the proof of Theorem 3, $f^w = f^{o^w}$ where for each $S \subset N$ and $i \in S$, $o_i^w(S) = \sum_{\pi \in \Pi_N} w_\pi o_i^\pi(S)$. For each $o \in SO$ we define $w_o = w_\pi$ if $o = o^\pi$ for some $\pi \in \Pi_N$ and $w_o = 0$ otherwise. Since $w \in \Delta(\Pi_N)$, $(w_o)_{o \in SO} \in \Delta(SO)$. Moreover, for each $S \subset N$ and $i \in S$, $o_i^w(S) = \sum_{o \in SO} w_o o_i(S)$.

Proof of " $\Leftarrow$ ": Let $f^{o'}$ be such that for each $S \subset N$ and $i \in S$, $o'_i(S) = \sum_{o \in SO} w_o o_i(S)$ and $(w_o)_{o \in SO} \in \Delta(SO)$.

For each $o \in SI$ we define the order $\pi^o \in \Pi_N$ as follows.

$$\pi^o(1) = \{i \in N : o_i(N) = 1\}.$$

In general, for each $k = 2, ..., |N|$,

$$\pi^o(k) = \left\{ \begin{array}{c} i \in N \setminus \{\pi^o(1), ..., \pi^o(k-1)\} : \\ o_i(N \setminus \{\pi^o(1), ..., \pi^o(k-1)\}) = 1 \end{array} \right\}.$$

Let $w = (w_\pi)_{\pi \in \Pi_N}$ be such that $w_\pi = w_{\pi^o}$ if $\pi = \pi^o$ for some $o \in SO$ and $w_\pi = 0$ otherwise. Since $(w_o)_{o \in SO} \in \Delta(SI)$, $(w_\pi)_{\pi \in \Pi_N} \in \Delta(\Pi_N)$. Hence, $f^w \in W$.

Let o^w be the obligation function associated with w as in the proof of STEP 1 of the proof of Theorem 3. It is trivial to see that $o^w = o'$.

Because of the proof of STEP 1 of the proof of Theorem 3, $f^w = f^{o^w}$. Therefore, $f^w = f^{o'}$.

8.5 Proof of Theorem 6.1

The proof of this theorem is a consequence of the following claims.

Claim 1. If f satisfies CPL, then f also satisfies $CSEC$.

Claim 2. If $f \in O$, then f satisfies SCM, PM, and CPL.

Claim 3. If f satisfies SCM, PM, and $CSEC$, then $f \in O$.

Proof of Claim 1. Let (N_0, C), (N_0, C'), (N_0, C^x), and (N_0, C'^x) be as in the definition of $CSEC$. We can find σ and σ' satisfying the following conditions:

1. $\sigma(0,i) = \sigma'(0,i) = \frac{n(n+1)}{2} - (i-1)$ for all $i \in N = \{1, ..., |N|\}$, *i.e.* the arcs $\{(0,i)\}_{i \in N}$ are the last arcs in the orders σ and σ'.
2. For all $i, j, k, l \in N_0$ satisfying that $\sigma(i,j) \leq \sigma(k,l)$, $c_{ij} \leq c_{kl}$ and $c_{ij}^x \leq c_{kl}^x$.
3. For all $i, j, k, l \in N_0$ satisfying that $\sigma'(i,j) \leq \sigma'(k,l)$, $c'_{ij} \leq c'_{kl}$ and $c'^x_{ij} \leq c'^x_{kl}$.

Consider the *mcstp* $\left(N_0, \widehat{C}\right)$ such that $\widehat{c_{0i}} = x$ for all $i \in N$ and $\widehat{c_{ij}} = 0$ otherwise. It is trivial to see that given $i, j, k, l \in N_0$ satisfying that $\sigma(i,j) \leq \sigma(k,l)$, $\widehat{c_{ij}} \leq \widehat{c_{kl}}$. Moreover, given $i, j, k, l \in N_0$ satisfying that $\sigma'(i,j) \leq \sigma'(k,l)$, $\widehat{c_{ij}} \leq \widehat{c_{kl}}$.

$C^x = C + C^e$ and $C'^x = C' + C^e$. Since f satisfies CPL,

$$\begin{aligned} f(N_0, C^x) &= f(N_0, C) + f\left(N_0, \widehat{C}\right) \text{ and} \\ f(N_0, C'^x) &= f(N_0, C') + f\left(N_0, \widehat{C}\right). \end{aligned}$$

Now, it is obvious that f satisfies $CSEC$.

This finishes the proof of Claim 1.

Proof of Claim 2. Tijs *et al* (2006) prove that obligation rules satisfy SCM and PM. The property called SCM in this paper is called cost monotonicity in Tijs *et al* (2006).

Bergantiños and Vidal-Puga (2006b) introduce the property of Restricted Additivity (RA). They prove that RA is stronger than CPL, *i.e.* if a rule f satisfies RA, f also satisfies CPL. Lorenzo-Freire and Lorenzo (2006) prove that obligation rules satisfy RA. Thus, obligation rules satisfy CPL.

This finishes the proof of Claim 2.

Proof of Claim 3. Let f be a rule satisfying SCM, PM, and $CSEC$.

We prove that $f = f^o$ where o is defined as follows. For each $S \subset N$ we define the *mcstp* $\left(S_0, C^S\right)$ where $c_{0i}^S = 1$ for each $i \in S$ and $c_{ij}^S = 0$ for each $i, j \in S$. Given $S \subset N$ and $i \in S$ we define $o_i(S) = f_i\left(S_0, C^S\right)$

We prove that o is an obligation function. Given $S \subset N$,

$$\sum_{i \in S} o_i(S) = \sum_{i \in S} f_i\left(S_0, C^S\right) = m\left(S_0, C^S\right) = 1.$$

Let $\left(S_0, C^0\right)$ be such that $c_{ij}^0 = 0$ for all $i, j \in S_0$.

We know that $f_i\left(\{i\}_0, C^0\right) = m\left(\{i\}_0, C^0\right) = 0$. Since f satisfies PM, $f_i\left(S_0, C^0\right) \leq f_i\left(\{i\}_0, C^0\right) = 0$ for each $i \in S$. As $m\left(S_0, C^0\right) = 0$, $f_i\left(S_0, C^0\right) = 0$ for each $i \in S$.

Since f satisfies SCM, for each $i \in S$,

$$o_i(S) = f_i\left(S_0, C^S\right) \geq f_i\left(S_0, C^0\right) = 0.$$

Thus, $o(S) \in \Delta(S)$.

Let $S, T \in 2^N \setminus \{\emptyset\}$ be such that $S \subset T$ and $i \in S$. Notice that $\left(S_0, C^T\right) = \left(S_0, C^S\right)$. Since f satisfies PM,

$$o_i(S) = f_i\left(S_0, C^S\right) = f_i\left(S_0, C^T\right) \geq f_i\left(T_0, C^T\right) = o_i(T)$$

We have proved that o is an obligation function. We now prove that $f = f^o$. Let (N_0, C) be an *mcstp*.

Bergantiños and Vidal-Puga (2006a) prove that if f satisfies SCM, then $f(N_0, C) = f(N_0, C^*)$. Thus, it is enough to prove that $f = f^o$ in irreducible *mcstp*.

Let (N_0, C) be an *mcstp* where C is irreducible. Let $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ be the *mt* in (N_0, C) satisfying $(A1)$ and $(A2)$

We prove that $f(N_0, C) = f^o(N_0, C)$ by induction over $|N|$. If $|N| = 1$, $f_i(N_0, C) = f_i^o(N_0, C) = c_{0i}$. Assume that the results holds when $|N| < \kappa$. We prove it when $|N| = \kappa$. We consider two cases:

1. $c_{0i_1} \leq \max\left\{c_{i_{p-1}i_p} : p = 2, ..., |N|\right\} = c_{i_{q-1}i_q}$.

We define $S = \{i_1, ..., i_{q-1}\}$. Since f and f^o satisfy PM, for each $i \in S$, $f_i(N_0, C) \leq f_i(S_0, C)$ and $f_i^o(N_0, C) \leq f_i^o(S_0, C)$. Moreover, for each $i \in N \setminus S$, $f_i(N_0, C) \leq f_i((N \setminus S)_0, C)$ and $f_i^o(N_0, C) \leq f_i^o((N \setminus S)_0, C)$.

Bergantiños and Vidal-Puga (2006a) prove that $\{(i_{p-1}, i_p)\}_{p=1}^{q-1}$ is an mt in (S_0, C) and $\{(0, i_q)\} \cup \{(i_{p-1}, i_p)\}_{p=q+1}^{|N|}$ is an mt in $((N \setminus S)_0, C)$. Thus, $m(S_0, C) = \sum_{p=1}^{q-1} c_{i_{p-1}i_p}$ and $m((N \setminus S)_0, C) = c_{0i_q} + \sum_{p=q+1}^{|N|} c_{i_{p-1}i_p}$. By $(A2)$, $c_{0i_q} = \max_{p \leq q}\{c_{i_{p-1}i_p}\} = c_{i_{q-1}i_q}$. Hence,

$$m(S_0, C) + m((N \setminus S)_0, C) = \sum_{p=1}^{|N|} c_{i_{p-1}i_p} = m(N_0, C)$$

Now it is easy to conclude that, for each $i \in S$, $f_i(N_0, C) = f_i(S_0, C)$ and $f_i^o(N_0, C) = f_i^o(S_0, C)$. Moreover, for each $i \in N \setminus S$, $f_i(N_0, C) = f_i((N \setminus S)_0, C)$ and $f_i^o(N_0, C) = f_i^o((N \setminus S)_0, C)$.

Since $i_1 \in S$ and $i_q \in N \setminus S$, $|S| < \kappa$ and $|N \setminus S| < \kappa$. By induction hypothesis, $f(S_0, C) = f^o(S_0, C)$ and $f((N \setminus S)_0, C) = f^o((N \setminus S)_0, C)$ Thus, $f(N_0, C) = f^o(N_0, C)$.

2. $c_{0i_1} > \max\{c_{i_{p-1}i_p} : p = 2, ..., |N|\} = c_{i_{q-1}i_q}$.

   We define the *mcstp* (N_0, C^0), (N_0, C^{0x}), (N_0, C^1), and (N_0, C^{1x}) where $x = c_{0i_1} - c_{i_{q-1}i_q}$ and

   - $c_{ij}^0 = 0$ for all $i, j \in N_0$.
   - $c_{0i}^{0x} = x$ for all $i \in N$ and $c_{ij}^{0x} = 0$ otherwise.
   - $c_{0i}^1 = c_{0i} - x$ for all $i \in N$ and $c_{ij}^1 = c_{ij}$ otherwise.
   - $C^{1x} = C$.

   Since f and f^o satisfy $CSEC$,

$$\begin{aligned} f(N_0, C^{1x}) - f(N_0, C^1) &= f(N_0, C^{0x}) - f(N_0, C^0) \text{ and} \\ f^o(N_0, C^{1x}) - f^o(N_0, C^1) &= f^o(N_0, C^{0x}) - f^o(N_0, C^0). \end{aligned}$$

   We proved above that $f_i(N_0, C^0) = 0$ for all $i \in N$. Moreover, $f_i^o(N_0, C^0) = 0$ for all $i \in N$.

   C^1 satisfies the following condition,

$$\begin{aligned} c_{0i_1}^1 &= c_{i_{q-1}i_q} = \max\{c_{i_{p-1}i_p} : p = 2, ..., |N|\} \\ &= \max\left\{c_{i_{p-1}i_p}^1 : p = 1, ..., |N|\right\} = c_{i_{q-1}i_q}^1. \end{aligned}$$

By Case 1, $f\left(N_0, C^1\right) = f^o\left(N_0, C^1\right)$ Thus,

$$\begin{aligned} f\left(N_0, C\right) &= f^o\left(N_0, C^1\right) + f\left(N_0, C^{0x}\right) \text{ and} \\ f^o\left(N_0, C\right) &= f^o\left(N_0, C^1\right) + f^o\left(N_0, C^{0x}\right). \end{aligned}$$

Since f^o satisfies CPL, $f^o\left(N_0, C^{0x}\right) = x f^o\left(N_0, C^N\right)$. Because of the definition of obligation rules, $f^o\left(N_0, C^N\right) = o\left(N\right)$. Thus,

$$f^o\left(N_0, C^{0x}\right) = x o\left(N\right) = x f\left(N_0, C^N\right).$$

We now prove that $f\left(N_0, C^{0x}\right) = x f\left(N_0, C^N\right)$ Assume that $x > 0$ is an integer number. Since f satisfies $CSEC$,

$$\begin{aligned} f(N_0, C^{0x}) - f(N_0, C^{0(x-1)}) &= f(N_0, C^{01}) - f(N_0, C^0) \\ &= f(N_0, C^{01}) = f\left(N_0, C^N\right). \end{aligned}$$

Hence, $f(N_0, C^{0x}) = f(N_0, C^{0(x-1)}) + f(N_0, C^N)$. By repeating this argument we get, $f(N_0, C^{0x}) = x f(N_0, C^N)$

Let x be a rational number. We can write $x = \frac{p}{q}$, where p and q are positive integers. Using similar arguments to those used before we can prove that $f(N_0, C^{0p}) = q f(N_0, C^{0x})$. Since p is an integer, we get $f(N_0, C^{0p}) = p f(N_0, C^N)$. Therefore,

$$f(N_0, C^{0x}) = \frac{1}{q} f(N_0, C^{0p}) = \frac{p}{q} f(N_0, C^N) = x f\left(N_0, C^N\right).$$

Let x be an irrational number. Consider $\{x_p\}_{p=1}^{\infty}$ a sequence of rational numbers satisfying that $x_p \leq x$ for all $p = 1, \ldots\infty$ and $\lim\limits_{p\to\infty} x_p = x$. By $CSEC$

$$f\left(N_0, C^{0x}\right) - f\left(N_0, C^{0x_p}\right) = f\left(N_0, C^{0(x-x_p)}\right) - f(N_0, C^0).$$

As $f_i(N_0, C^0) = 0$ for all $i \in N$,

$$f\left(N_0, C^{0x}\right) = f\left(N_0, C^{0x_p}\right) + f\left(N_0, C^{0(x-x_p)}\right).$$

Since $C^{0(x-x_p)} \geq C^0$ and f satisfies SCM, $f_i\left(N_0, C^{0(x-x_p)}\right) \geq f_i(N_0, C^0) = 0$ for each $i \in N$.

As $\sum\limits_{i \in N} f_i\left(N_0, C^{0(x-x_p)}\right) = m\left(N_0, C^{0(x-x_p)}\right) = x - x_p$, $f_i\left(N_0, C^{0(x-x_p)}\right) \leq x - x_p$ for each $i \in N$.

Therefore, for each $i \in N$, $0 \leq f_i\left(N_0, C^{0(x-x_p)}\right) \leq x - x_p$. Hence,

$$0 \leq \lim_{p\to\infty} f_i\left(N_0, C^{0(x-x_p)}\right) \leq \lim_{p\to\infty} \left(x - x_p\right) = 0.$$

Moreover,

$$\lim_{p\to\infty} f\left(N_0, C^{0x_p}\right) = \lim_{p\to\infty} x_p f\left(N_0, C^N\right) = xf\left(N_0, C^N\right).$$

Thus,

$$\begin{aligned} f\left(N_0, C^{0x}\right) &= \lim_{p\to\infty} f\left(N_0, C^{0x_p}\right) + \lim_{p\to\infty} f\left(N_0, C^{0(x-x_p)}\right) \\ &= xf\left(N_0, C^N\right). \end{aligned}$$

This ends the proof of Claim 3.

8.6 Proof of Remark 6.1

- The rule $\phi(N_0, C) = Sh(N, v_C)$ satisfies CPL, $CSEC$ and PM but fails SCM.
- The equal division rule (that is $\phi_i(N_0, C) = \frac{m(N_0,C)}{|N|}$) satisfies SCM, CPL, and $CSEC$ but fails PM.
- For a rule that satisfies SCM and PM but fails to satisfy $CSEC$, see the companion paper Bergantiños and Kar (2007b).

8.7 Proof of Corollary 6.1

It is easy to check that Φ satisfies ETE. Since $\Phi \in W \subseteq O$, Φ is an obligation rule and hence SCM, PM, and $CSEC$ are satisfied. We now show that there is only one rule which satisfies these properties. The proof is very similar to the one used in Theorem 6.1.

Let (N_0, C) be an $mcstp$ where C is irreducible. Let $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ be the mt in (N_0, C) satisfying $(A1)$ and $(A2)$.

We use induction over $|N|$. If $|N| = 1$, $f_i(N_0, C) = c_{0i}$ and hence there is only one rule. Assume that the result holds when $|N| < \kappa$. We prove it when $|N| = \kappa$. We consider two cases:

1. $c_{0i_1} \leq \max\left\{c_{i_{p-1}i_p} : p = 2, ..., |N|\right\} = c_{i_{q-1}i_q}$. This case is exactly the same as in the proof of Theorem 6.1 and hence will be omitted. In this case using induction step we get that there is only one rule f which satisfies all the axioms.
2. $c_{0i_1} > \max\left\{c_{i_{p-1}i_p} : p = 2, ..., |N|\right\} = c_{i_{q-1}i_q}$.

   We define the $mcstp$ $\left(N_0, C^0\right)$, $\left(N_0, C^{0x}\right)$, $\left(N_0, C^1\right)$, and $\left(N_0, C^{1x}\right)$ where $x = c_{0i_1} - c_{i_{q-1}i_q}$ and

   - $c_{ij}^0 = 0$ for all $i, j \in N_0$.

- $c_{0i}^{0x} = x$ for all $i \in N$ and $c_{ij}^{0x} = 0$ otherwise.
- $c_{0i}^{1} = c_{0i} - x$ for all $i \in N$ and $c_{ij}^{1} = c_{ij}$ otherwise.
- $C^{1x} = C$.

Since f satisfies $CSEC$,

$$f_i\left(N_0, C^{1x}\right) - f_i\left(N_0, C^{1}\right) = f_i\left(N_0, C^{0x}\right) - f_i\left(N_0, C^{0}\right) = \frac{x}{\kappa}$$

The last equality follows from the fact that in C^0 and C^{0x} all agents are identical and f satisfies ETE. Thus for all $i \in N$; $f_i\left(N_0, C\right) = f_i\left(N_0, C^1\right) + \frac{x}{\kappa}$. However C^1 is a cost matrix which is already covered in case 1 and we know $f(N_0, C^1)$ is unique. Therefore $f(N_0, C)$ is also unique.

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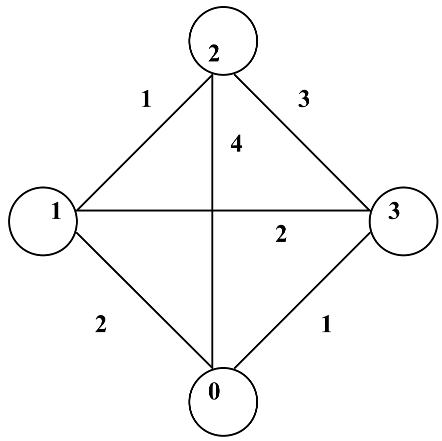

Figure 1

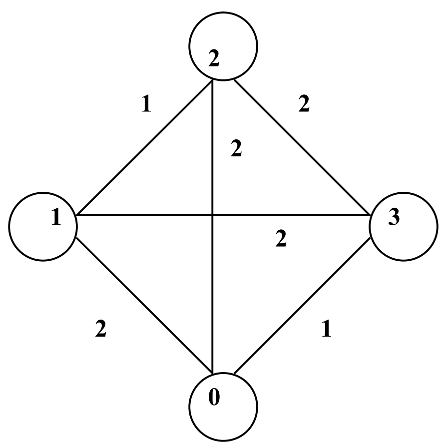

Figure 2